GRAPHICAL METHOD

FOR FINDING READILY THE

REAL ROOTS

OF

NUMERICAL EQUATIONS OF ANY DEGREE

IF CONTAINING BUT

ONE VARIABLE.

BY

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PREFACE.

THIS graphical method is due to Captain Lill, of the Austrian service, who first exhibited it at the Paris Exposition of 1867.

The method is now brought forward, by the undersigned, because he is not aware it has yet been presented to the English-reading public; and because it possesses some novel features which may assist in enlarging the already broad field of graphical analysis.

The proof of the correctness of the method, as given in the following pages, is, as far as I know, original with myself.

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WEST POINT, *March*, 1879.

GRAPHIC SOLUTION OF NUMERICAL EQUATIONS OF ANY DEGREE, IF CONTAINING BUT ONE VARIABLE.

[N. B. Those readers who care nothing about the proof, but who desire to consult only the application, of the following constructions, will save time by skipping immediately to p. 13].

There are two geometrical propositions so much used, in the following pages, that their solution will precede that of the main problem.

They are given in the following lemmas.

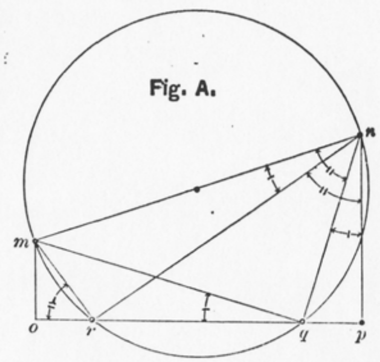
Lemma I.

Let *mn* (Fig. A) be any diameter, and *rq* any chord of a circle; let *mo* and *np* be the perpendiculars let fall from the extremities of the diameter, upon the chord produced : then ;

1*st*, $\angle mnp = \angle mnq + \angle mnr$, and

2*nd*, $pq = ro$.

Fig. A.



First ; $\angle rqn = \angle qpn + \angle pnq = 90^\circ + \angle pnq$; then $\angle pnq = \angle mqr = \angle mnr$, and hence $\angle mnp = \angle mnq + \angle qnp = \angle mnq + \angle mnr$. Q. E. D.

Second ; since $\angle pnq = \angle mqr = \angle mnr$, then $\angle mnq = \angle rnp = \angle mro$, and cot. $\angle pnq =$ cot. $\angle mqr$; therefore, $\dfrac{pn}{pq} = \dfrac{qo}{mo} = \dfrac{qr + ro}{mo}$. (*a*)

Also, cot. $\angle rnp =$ cot. $\angle mro$; therefore $\dfrac{np}{pr} = \dfrac{np}{pq+qr} = \dfrac{ro}{mo}$. (*b*)

From (*a*) we have $(pn)\,(mo) = (qr+ro)\,pq$, and from (*b*)

$$(pn)\,(mo) = (pq+qr)\,ro\,;$$

whence combining and reducing $(pq)\,(qr) = (qr)\,(ro)$ from which $pq = ro$.

Q. E. D.

Lemma II.

Let mn (Fig. B) be any diameter, and qr any chord of a circle; from m and n drop perpendicular mo and np upon qr produced. Let $qr_{,}$ be any other chord through q, and let $mo_{,}$ and $np_{,}$ be the perpendiculars let fall upon $qr_{,}$ from m and n respectively; then the straight line through $oo_{,}$ will be perpendicular to the straight line through $pp_{,}$.

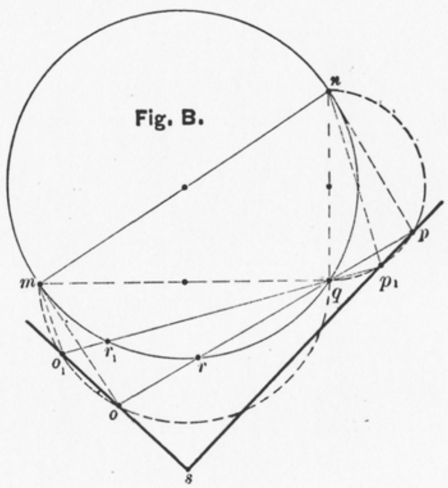


Fig. B.

For, since $\angle mo_{,}q = 90° = \angle moq$, then o and $o_{,}$ are on the circle whose diameter is mq; and since $\angle npq = 90° = \angle np_{,}q$. p and $p_{,}$ are on the circle whose diameter is nq.

Let s be the intersection of $oo_{,}$ with $pp_{,}$; then $osp = 180° - \begin{pmatrix} + \angle spq \\ + \angle qos \end{pmatrix}$

$$= 180° - \begin{pmatrix} + \angle p_{,}nq \\ + \angle o_{,}mq \end{pmatrix} = 180° - \begin{pmatrix} +90° - \angle nqp_{,} \\ +90° - \angle mqo_{,} \end{pmatrix} = \angle nqp_{,} + \angle mqo_{,}$$

$$= 180° - \angle nqm = 90°.$$

Hence, $oo_{,}$ is perpendicular to $pp_{,}$. Q. E. D.

Equations of the First Degree.

Assume (Fig. 1) any two points α and ω; join them by a straight line; upon this line describe a circumference; select on this circumference any point a; represent the angle $\alpha\,\omega a$ by γ, the distance $\alpha\, a$ by A, $a\omega$ by B; and we shall then

have $\frac{\omega a}{a\alpha} = \frac{B}{A} = \cot.\,\gamma =$

the cotangent of the angle whose vertex is at ω and whose sides pass through the starting point α and the assumed point a.

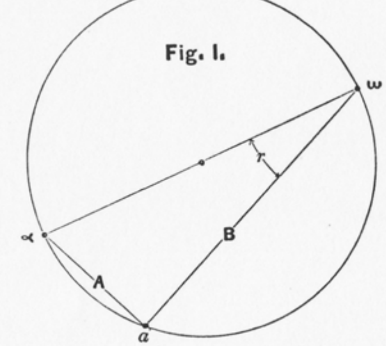


Fig. I.

Now, if we have any equation of the first degree, containing but one variable, it can be placed under the general form of $A\,x + B = o$,

whose root is evidently $x = -\frac{B}{A}$.

Therefore, if by any convenient scale, we lay off a distance $\alpha\, a$ equal to A, draw through a a perpendicular to $\alpha\, a$, and lay off $a\omega$ equal to B, (positive distances being measured from a towards ω in the direction employed in Fig. 1) we shall evidently have $\frac{a\omega}{\alpha\, a} = \frac{B}{A} = -$(the root of the given equation); and the rectangular contour* $\alpha\, a\omega = AB$ may be taken as representing an equation $A\,x + B = o$, whose root, taken with its sign changed, is $\frac{a\omega}{\alpha\, a} = \frac{B}{A}$.

Equations of the Second Degree.

Let α, a, and ω, (Fig. 2), represent the same quantities as in Fig. 1; let b be any other point on the circumference $\alpha\, a\omega$; through ab draw a straight line; from α and ω, let fall the perpendiculars $\alpha\, a_{,}''$ and $\omega b_{,}''$, upon the line ab; represent $\alpha\, a$ by A, $a\omega$ by B, $\alpha\, b$ by A', and $b\omega$ by B', the $\angle \alpha\, \omega a$ by γ, $\angle \alpha\, \omega b$ by γ'.

Then $\cot.\,\gamma = \frac{B}{A}$, and

$\cot.\,\gamma' = \frac{B'}{A'}$. (1)

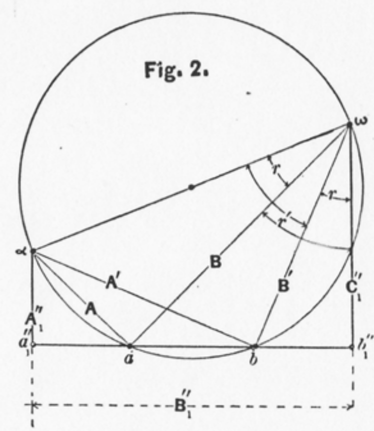


Fig. 2.

From Lemma I, we have $\angle b_{,}''\omega b = \angle \alpha\, \omega a = \gamma$, and $\angle b_{,}''\omega a = \angle \alpha\, \omega b = \gamma'$; also (Lemma I) $\left.\begin{matrix} b_{,}''b = aa_{,}'' \\ b_{,}''a = ba_{,}'' \end{matrix}\right\}$ (2)

* By "rectangular contour" is meant a broken line in which each straight portion is perpendicular to each of the adjacent portions.

From similarity of right-angled triangles,

$$\frac{a_,''b}{\propto a_,''} = \frac{b_,''\omega}{b\,b_,''} \text{ and } \frac{a_,''a}{\propto a_,''} = \frac{b_,''\omega}{a\,b_,''},$$

$$\left.\begin{aligned} \text{whence cot. } \gamma &= \frac{\text{B}}{\text{A}} = \frac{a\omega}{\propto a} = \frac{b_,''\omega}{b\,b_,''} = \frac{a_,''b}{\propto a_,''} \\ \text{cot. } \gamma' &= \frac{\text{B}'}{\text{A}'} = \frac{b\omega}{\propto b} = \frac{b_,''\omega}{a\,b_,''} = \frac{a_,''a}{\propto a_,''} \end{aligned}\right\} \quad (3)$$

Represent the diameter $\propto \omega$ by δ;

$$\left.\begin{aligned} \text{then} \quad \text{A} &= \delta \sin. \gamma \quad \text{whence } \sin. \gamma = \frac{\text{A}}{\delta} \\ \text{A}' &= \delta \sin. \gamma' \qquad \sin. \gamma' = \frac{\text{A}'}{\delta} \end{aligned}\right\} \quad (4)$$

But $\quad \propto a_,'' = \text{A} \sin. \gamma' = \dfrac{\text{AA}'}{\delta}$.

Also, $\quad a_,''\,a = bb_,'' = \text{B}' \sin. \gamma = \dfrac{\text{B}'\text{A}}{\delta}$

$$ab_,'' = \text{B} \sin. \gamma' = \frac{\text{BA}'}{\delta},$$

whence $\quad a_,''\,b'' = \dfrac{\text{B}'\text{A} + \text{BA}'}{\delta}$.

Also, $\quad \text{B} = \delta \cos. \gamma, \qquad$ whence $b_,''\,\omega = \text{B}' \cos. \gamma = \dfrac{\text{BB}'}{\delta}$.

Denote $\propto a_,''$ by $\text{A}_,''$, $a_,''\,b_,''$ by $\text{B}_,''$, $b_,''\,\omega$ by $\text{C}_,''$, and we have the rectangular contour $\text{A}_,''\,\text{B}_,''\,\text{C}_,''$, whose sides

$$\left.\begin{aligned} &\text{A}_,'' = \frac{\text{AA}'}{\delta},\ \text{B}_,'' = \frac{\text{B}'\text{A}+\text{BA}'}{\delta}, \text{ and } \text{C}_,'' = \frac{\text{BB}'}{\delta}, \\ &\text{and in which (Eq. 3)} \quad \frac{a_,''b}{\text{A}_,''} = \frac{\text{B}}{\text{A}} \text{ and } \frac{a_,''a}{\text{A}_,''} = \frac{\text{B}'}{\text{A}'}. \end{aligned}\right\} \quad (5)$$

Now, if we have any equation of the second degree, containing but one variable, it can be placed under the general form of $\text{A}_,''\ x^2 + \text{B}_,''\ x + \text{C}_,'' = 0$, whose roots will be $x = -\dfrac{\text{B}}{\text{A}}$, $x = -\dfrac{\text{B}'}{\text{A}'}$, if $\text{A}_,'' = \text{AA}'$, $\text{B}_,'' = \text{AB}' + \text{A}'\text{B}$, and $\text{C}_,'' = \text{BB}'$.

Therefore, from any point $\propto$, and with any convenient scale, lay off (Fig. 2) $\propto a_,'' = \text{A}_,''$; then perpendicular to $\text{A}_,$, from its extremity, lay off $a_,''\ b_,'' = \text{B}_,''$ (positive distances being measured from $a_,''$ towards $b_,''$ with the direction used in Fig. 2); and, in like manner, lay off $b_,''\ \omega = \text{C}_,''$; and we shall then have a three-sided rectangular contour which may be regarded as representing the equation

$$[\text{A}_,''\ x^2 + \text{B}_,''\ x + \text{C}_,'' = o] = \left[\frac{\text{AA}'\ x^2 + (\text{AB}' + \text{B}'\text{A})\ x + \text{BB}'}{\delta} = o\right]$$

$$= \left[(\text{A}\ x + \text{B} = o)(\text{A}'\ x + \text{B}' = o)\left(\tfrac{1}{\delta}\right) = o\right]$$

where $\text{A}\ x + \text{B} = o$, and $\text{A}'\ x + \text{B}' = o$, are represented respectively by the two-sided rectangular contours $\propto a\omega$ and $\propto b\omega$, all of the vertices, in both cases, resting upon the three-sided rectangular contour $\text{A}_,''\ \text{B}_,''\ \text{C}_,''$; the roots of the given equation being equal, with changed sign, to $\dfrac{1}{\text{A}''}$ multiplied into that portion of $\text{B}_,''$ which lies between $\text{A}_,''$ and the lines A and A′ respectively.

N. B. The scale of the contour $\text{A}_,''\ \text{B}_,''\ \text{C}_,''$ is evidently not the same as the scale of the contours AB and A′ B′.

Equations of the Third Degree.

Assume (Fig. 3) the points $\propto$, ω, a, and b, as in Fig. 2; represent, as before, $\propto a$ by A, $\propto b$ by A′, ωa by B, ωb by B′; construct as before $\propto a_,''$ and $\omega b_,''$, representing

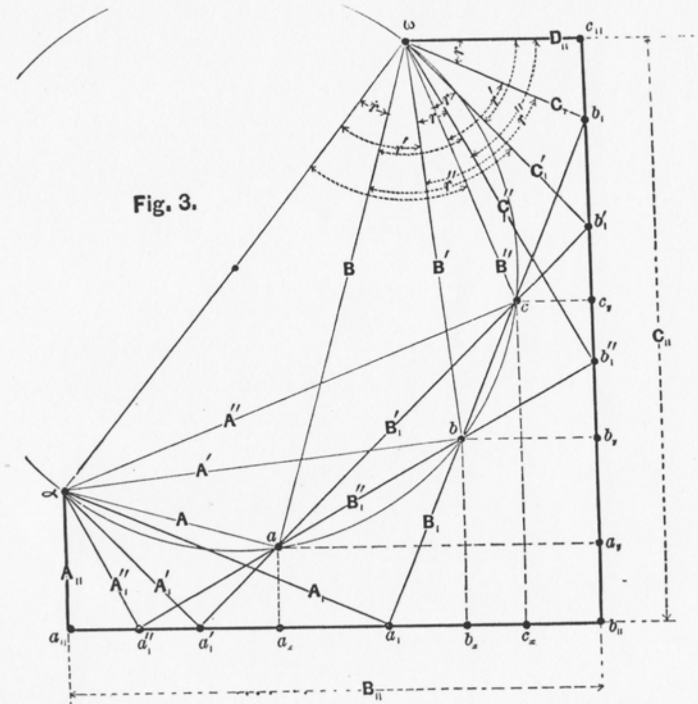


Fig. 3.

$\propto a_,''$ by $\text{A}_,''$, $a_,''\,b_,''$ by $\text{B}_,''$, $b_,''\,\omega$ by $\text{C}_,''$. Let c be any other point on the circumference $\propto ab\omega$.

Treating the points b and c as we have, in Fig. 2, treated a and b, we shall have $\propto c = \text{A}''$, $\omega c = \text{B}''$, $\angle \propto \omega c = \gamma'' = \angle b\omega b_, = \angle \propto ba_,$. $\gamma' = \angle c\omega b_, = \angle \propto ca_,$, and we shall obtain the three-sided rectangular contour $\propto a_,\ b_,\ \omega = \text{A}_,\ \text{B}_,\ \text{C}_,$ in which $a_, b = cb_,$, and $a_, c = bb_,$.

Consider the two contours $\text{A}_,\ \text{B}_,\ \text{C}_,$ and $\text{A}_,''\ \text{B}_,''\ \text{C}_,''$; the right line $b_,\,b_,''$ is perpendicular (Lemma II) to the right line $a_,\,a_,''$ at some point as $b_{,,}$; let fall from $\propto$ and ω, new perpendiculars $\propto a_{,,}$ and $\omega c_{,,}$ upon $a_,\,a_,''$ and $b_,\,b_,''$ respectively, denoting $\propto a_{,,}$ by $\text{A}_{,,}$, $a_{,,}\,b_{,,}$ by $\text{B}_{,,}$, $b_{,,}\,c_{,,}$ by $\text{C}_{,,}$, and $c_{,,}\,\omega$ by $\text{D}_{,,}$; we shall now have a new four-sided rectangular contour $\propto a_{,,}\,b_{,,}\,c_{,,}\,\omega = \text{A}_{,,}\ \text{B}_{,,}\ \text{C}_{,,}\ \text{D}_{,,}$, whose sides contain the vertices of both the three-sided contours $\text{A}_,\ \text{B}_,\ \text{C}_,$ and $\text{A}_,''\ \text{B}_,''\ \text{C}_,''$.

Since $\angle bb_,''\ \omega = 90° = \angle bb_,\ \omega$, we have (Lemma I) $\angle b_,\,\omega c_{,,} = b\omega b_,'' = \gamma$, and $\angle \propto \omega c_{,,} = \angle \propto \omega c + \angle c\omega b_, + \angle b_,\omega c_{,,} = \gamma'' + \gamma' + \gamma$.

Considering the points a and c, we shall in like manner get another three-sided rectangular contour $\propto a_,'\,b_,'\ c\ \ \omega = \text{A}_,'\ \text{B}_,'\ \text{C}_,'$; combining contours $\text{A}_,'\ \text{B}_,'\ \text{C}_,'$ and

A, B, C, we get a four-sided contour whose fourth side as before must make with the line $\propto \omega$, an angle equal to $\gamma + \gamma' + \gamma''$; this fourth side must then coincide in direction with $\omega c_{,,}$; the third side must contain the point $b_{,}$ and be perpendicular to the fourth side, and hence must coincide with $c_{,} b_{,,}$; for a similar reason the second side must coincide with $b_{,,} a_{,,}$, the first side with $a_{,,} \propto$.

Combining the contours $A_{,}' B_{,}' C_{,}'$ and $A_{,}'' B_{,}'' C_{,}''$ we get a similar result.

Hence the three three-sided rectangular contours $A_{,} B_{,} C_{,}$, $A_{,}' B_{,}' C_{,}'$, and $A_{,}'' B_{,}'' C_{,}''$, have all their vertices resting upon the sides of a single four-sided rectangular contour $\propto a_{,,} b_{,,} c_{,,} \omega = A_{,,} B_{,,} C_{,,} D_{,,}$; all four contours commencing at $\propto$ and ending at ω.

From similar triangles,

$$\left.\begin{array}{l} \angle \propto a_{,}'' a_{,,} = \angle a_{,}'' b_{,}'' b_{,,} = \angle b_{,}'' \omega c_{,,} = \\ \text{(Lemma I)} \angle b \omega b_{,} = \text{(Lemma I)} \angle \propto \omega c = \gamma'' \end{array}\right\} \quad (6)$$

and in like manner

$$\left.\begin{array}{l} \angle \propto a_{,}' a_{,,} = \angle a_{,}' b_{,}' b_{,,} = \angle b_{,}' \omega c_{,,} \\ \quad = \angle c \omega b_{,} = \angle \propto \omega b = \gamma' \end{array}\right\} \quad (7)$$

and

$$\left.\begin{array}{l} \angle \propto a_{,} a_{,,} = \angle a_{,} b_{,} b_{,,} = \angle b_{,} \omega c_{,,} \\ \quad = \angle c \omega b_{,}' = \angle \propto \omega a = \gamma \end{array}\right\} \quad (8)$$

From triangles $\propto a_{,,} a_{,}''$, $A_{,,} = A_{,}'' \sin \gamma''$, but (5) $A_{,}'' = \frac{AA'}{\delta}$, and (4)

$$\sin. \gamma'' = \frac{A''}{\delta} ; \text{ hence } A_{,,} = \frac{AA'}{\delta} \cdot \frac{A''}{\delta} = \frac{AA'\,A''}{\delta^2} \quad (9)$$

Lemma II. applied to points $\propto a_{,}'' a_{,}' a$ gives $a_{,}'' a x = a_{,,} a_{,}'$; (10)
applied to $\propto a_{,}'' a_{,} b$, gives $a_{,,} a_{,} = a_{,}'' b x$; (11)
applied to $\propto a b \omega$, gives $a_{,}'' b = a b_{,}''$; whence, by similar triangles,
$a_{,}'' b x ; = a x\, b_{,,}$ (12) whence (11) (12) $a x b_{,,} = a_{,,} a_{,}$. (13)

But

$$\left.\begin{array}{lllll} \frac{B''}{A''} = \frac{a'' a_{,}''}{\propto a''} & & = \frac{a_{,,} a_{,}''}{A_{,,}} & & \\ \frac{B'}{A'} = \frac{a_{,,} a_{,}'}{\propto a_{,,}} & = \text{(Eq. 10)} & \frac{a_{,}'' a x}{\propto a_{,,}} & = & \frac{a_{,}'' a x}{A_{,,}} \\ \frac{B}{A} = \frac{a_{,,} a_{,}}{\propto a_{,,}} & = \text{(Eq. 13)} & \frac{a x\, b_{,,}}{\propto a_{,,}} & = & \frac{a x\, b_{,,}}{A_{,,}} \end{array}\right\} \quad (14)$$

and adding equations (14) member to member, we get

$$\frac{B''}{A''} + \frac{B'}{A'} + \frac{B}{A} , = \frac{B_{,,}}{A_{,,}} \text{ and (9), } B'' = \left(\frac{B''}{A''} + \frac{B'}{A'} + \frac{B}{A}\right) \frac{AA'\,A''}{\delta^2} \quad (15)$$

$$\text{Now, } D_{,,} = C_{,}'' \cos. \gamma'' = B' \cos. \gamma \frac{B''}{\delta} = B' . \frac{B}{\delta} . \frac{B''}{\delta} ,$$

hence

$$D_{,,} = \frac{BB'\,B''}{\delta^2} \quad (16)$$

Treating $c_{,,} b_{,,}$ as we have just treated $a_{,,} b_{,,}$, we find $c_{,,} b_{,}' = b_{,} c y$, and $c y\, b_{,,} = c_{,,} b_{,}''$, and $\frac{c_{,,} b_{,}}{\omega\, c_{,,}} = \frac{c_{,,} b_{,}}{D} = \tan. \gamma = \frac{A}{B}$,

$$\frac{b_{,} c y}{\omega\, c_{,,}} = \frac{c_{,,} b_{,}'}{D_{,,}} = \text{tang. } \gamma' = \frac{A'}{B'} ,$$

$$\frac{c y\, b_{,,}}{\omega\, c_{,,}} = \frac{c_{,,} b_{,}''}{D_{,,}} = \text{tang. } \gamma'' = \frac{A''}{B''} , \text{ whence, adding,}$$

$$C_{,,} = D_{,,} \left(\frac{A}{B} + \frac{A'}{B'} + \frac{A''}{B''}\right) = \left(\frac{A}{B} + \frac{A'}{B'} + \frac{A''}{B''}\right) \frac{BB'\,B''}{\delta^2} . \quad (17)$$

Now, if we have any equation of the third degree, containing but one variable, it can be placed under the general form of $A_{,,} x^3 + B_{,,} x^2 + C_{,,} x + D_{,,} = 0$ whose roots will be $x = -\frac{B}{A}$, $x = -\frac{B'}{A'}$, $x = -\frac{B''}{A''}$, if $A_{,,} = AA'\,A''$, $B_{,,} = AA'\,B'' + AB'\,A'' + BA'\,A''$, $C_{,,} = AB'\,B'' + BA'\,B'' + BB'\,A''$, and $D_{,,} = BB'\,B''$.

Therefore, from any point $\propto$, and with any convenient scale, lay off (Fig. 3) $\propto a_{,,} = A_{,,}$; from the extremity of, and perpendicular to, $A_{,,}$, lay off $a_{,,} b_{,,} = B_{,,}$ (positive distance being measured from $a_{,,}$ towards $b_{,,}$ in the direction employed in Fig. 3); in like manner, lay off $b_{,,} c_{,,} = C_{,,}$, $c_{,,} \omega = D_{,,}$, and we shall then have a four-sided rectangular contour which may be regarded as representing the equation $[A_{,,} x^3 + B_{,,} x^2 + C_{,,} x + D_{,,} = 0]$ equal (Eq. 9, 15, 16, 17) to $\left[AA'\,A''\,x^3 + (AA'\,B'' + AB'\,A'' + BA'\,A'')\,x^2 + (AB'\,B'' + BA'\,B'' + BB'\,A'')\,x + BB'\,B''\right]\left(\frac{1}{\delta^2}\right)$ $= 0 = [(A x + B = 0)\,(A' x + B' = 0)\,(A'' x + B'' = 0)]$ where the three-sided rectangular contours $A_{,} B_{,} C_{,}$, $A_{,}' B_{,}' C_{,}'$, $A_{,}'' B_{,}'' C_{,}''$, represent respectively $[(A' x + B' = 0)\,(A'' x + B'' = 0)] = 0$, $[(A x + B = 0)\,(A'' x + B'' x = 0)] = 0$, and $[(A x + B = 0)\,(A' x + B' = 0)] = 0$; the vertices in all three cases resting upon the contour $A_{,,} B_{,,} C_{,,} D_{,,}$; the roots of the given equation being equal, with changed sign, to $\frac{1}{A_{,,}}$ multiplied into that portion of $B_{,,}$ which lies between $A_{,,}$ and the straight lines $A_{,}$, $A_{,}'$ and $A_{,}''$ respectively.

N. B. The scale of the contour $A_{,,} B_{,,} C_{,,} D_{,,}$ is different from the scale of $A_{,} B_{,} C_{,}$, $A_{,}' B_{,}' C_{,}'$, $A_{,}'' B_{,}'' C_{,}''$, and also from the scale of AB , A′ B′ , A″ B″ .

Equations of higher than the Third Degree.

The preceding method can be readily extended to include numerical equations of any degree, if containing but one variable.

For equations of the fourth degree, we shall find one contour of five sides, four of four sides, six of three sides, four of two sides, all resting upon the ends of the single line $\propto \omega$.

For equations of the fifth degree, we shall find one contour of six sides, five of five sides, ten of four sides, ten of three sides, five of two sides, and one (the diameter $\propto \omega$) of one side.

For equations of still higher degrees the number of the resulting contours evidently correspond to the terms of the series.

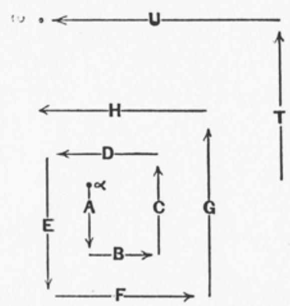
$$1, \quad n, \quad \frac{n(n-1)}{1.2}, \quad \frac{n(n-1)(n-2)}{1.2.3} \quad \ldots \quad \frac{n(n-1)(n-2)\ldots\ldots 3.2.1.}{1.2.3\ldots(n-2)(n-1)n}.$$

In theory this method is perfect for real roots; and it will indicate the presence and number of imaginary roots.

In practice, the method is rigorous only for equations of the second degree; in equations of higher degrees it gives only approximate results; and, in general practice, it is limited, for obvious reasons, to those real roots which lie between 0 and ± 4.0, within which limits it has however a wide and ready application.

APPLICATIONS.

Fig. 4.



Assume a broken line, as in Fig. 4, all the angles being right angles, lettering the straight portions from its commencement, in order, A B C D E F G, etc.; place upon these lines arrow-heads (as in the figure) to show the *directions in which distances* measured upon these lines A B C, etc., *are* to be regarded as *positive.*

Then any distance measured on A or E will be positive, if measured downwards; upon B and F, positive, if measured from left to right; on C and G, positive, if measured upward; on D and H, positive, if measured from right to left: and so on. Negative values of A B C D, etc., must always be measured in exactly the opposite directions to those just given.

Now suppose any numerical equation containing but one variable x; reduce it to the general form,

$$A_n x^n + B_n x^{n-1} + C_n x^{n-2} +, \text{ etc. } \ldots\ldots + T_n x + U_n = o,$$

in which A_n is a positive number, either whole or fractional.

Commence, on a blank sheet of paper, at any assumed point $\propto$, and using any convenient scale, lay off in a downward direction a distance $\propto a$ equal to A_n; through a, draw a perpendicular, and lay off upon it with the same scale as before, the value of B_n (laying off this distance to the right if B_n is positive, to the left, if B_n is negative); through the end of B_n, draw a perpendicular to B_n, upon which lay off the value of C_n, upward if positive, downward if negative; and so on, laying off positive distances always in the directions indicated by the arrow-heads of Fig. 5. Letter the end of the last line ω. We will then have a rectangular contour (that is, a broken line all of whose angles are right angles) of $n + 1$ sides commencing at $\propto$ and ending at ω.

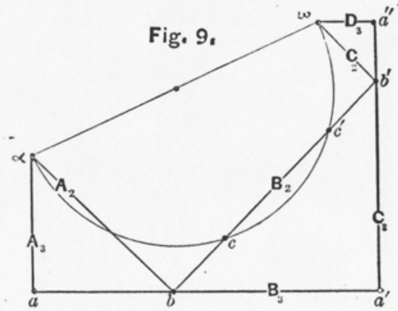
Now starting again at $\propto$, draw at random any straight line cutting B_n in some point as b; through b draw a perpendicular to $\propto b$ cutting C_n in some point as b', and so on; the result will be a new rectangular

Equations of the Third Degrèe.

Reduce the equation to the general form

$$A_3 x^3 + B_3 x^2 + C_3 x + D_3 = 0 .$$

Fig. 9.



The solution is here approximate.

Lay off (Fig. 9) $\propto a = A_3$, $aa' = B_3$, $a' a'' = C_3$, and $a'' \omega = D_3$, following the rule of signs indicated in Fig. 4.

By trial, determine the rectangular contour $A_2\ B_2\ C_2$, commencing at $\propto$ and ending at ω ;

then $\frac{ab}{\propto a}$ is one root of the given equation.

On $\propto \omega$, describe a circumference ; if it cuts B_2 in points c and c' . the other two roots of the given equation are real and equal to $\frac{bc}{\propto b}$ and $\frac{bc'}{\propto b}$; if the circumference on $\propto \omega$ touches B_2 at a point $c = c'$, then the other two roots of the given equation are real, equal, and equal to $\frac{bc}{\propto b} = \frac{bc'}{\propto b}$; if the circumference $\propto \omega$ has no point in common with B_2 , then the other two roots of the given equation are imaginary.

Equations of higher than the Third Degree.

Ordinarily most of the real roots can be readily found by the use of the tracing sheet and ruled paper, without drawing the auxiliary contours, simply following these contours by eye, turning the tracing slightly from time to time.

For example : suppose one auxiliary contour to be followed around by eye till it comes out upon the last portion of the first contour between the preceding line and ω ; turn the tracing sheet a little and trace a second contour by eye : if the latter comes out on the same side of ω as before, but nearer to ω, then continue turning ; if the new contour ends upon the opposite side of ω, then turn back, for between the last two contours will probably lie one which will give one of the desired roots.

After the solution of a few problems the reader will become quite ready in the application of this graphical method.

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