Fermat's Integration of Powers

The 1657 publication of the *Arithmetica infinitorum* (Arithmetic of Infinities) of John Wallis (1616–1703) prompted Pierre de Fermat (1601–1665) to compose a treatise on the quadrature methods he had been developing for two decades. The result was his posthumous *De aequationum localium transmutatione et emendatione* ... (On the transformation and alteration of local equations for the purpose of variously comparing curvilinear figures among themselves or to rectilinear figures, to which is attached the use of geometric proportions in squaring an infinite number of parabolas and hyperbolas). This was probably written in 1658, but not published (or circulated) until 1679 in his *Varia opera mathematica*, so it appeared too late to have a profound effect on the development of the calculus. Our interest is in the section dealing with the quadrature (or finding the area under) the "higher hyperbolas," $x^p y^q = k$, and "higher parabolas," $y^p = kx^q$, which Fermat created.

Fermat's ingenious integration of $y = x^n$ provides a very interesting example which can easily and profitably be done in class. When introducing the integral via the notion of Riemann sums, the problems quickly become too hard. Here is a neat trick that will allow you to use the definition to evaluate the integrals of powers of x in class—and for an arbitrary integer n. The only fact that is needed is the sum of the geometric series, a fact the student needs in other situations anyway.

The clever idea that Fermat had was not to divide the interval $[0, a]$ of integration into equal subdivisions, but rather to use unequal subdivisions. It is clear where this idea came from. He had been finding the area under his generalized hyperbolas $y = 1/x^n$ on the interval $[1, \infty]$, and in this situation it is natural to use equal subdivisions. But when he considered the generalized parabolas $y = x^n$ on $[0, 1]$, it was natural to invert the ray $[1, \infty]$ into the finite interval $[0, 1]$. When this is done, unequal partitions are the reasonable choice to make. So now let's see what Fermat did.

Let E be a positive constant less than 1. Use it to divide the closed interval $[0, a]$ into infinitely many subintervals of different lengths at the points $\ldots, aE^3, aE^2, aE, a$. Then construct rectangles on these subintervals so that they circumscribe the curve $y = x^n$ and add up their areas, which form a geometric progression:

$$\begin{aligned}
S_E &= \sum_{i=0}^{\infty} (aE^i)^n (aE^i - aE^{i+1}) \\
&= a^{n+1} \sum_{i=0}^{\infty} E^{in} E^i (1 - E) \\
&= a^{n+1}(1 - E) \sum_{i=0}^{\infty} (E^{n+1})^i \\
&= a^{n+1}(1 - E) \frac{1}{1 - E^{n+1}} \\
&= \frac{a^{n+1}}{1 + E + E^2 + E^3 + \ldots + E^n}.
\end{aligned}$$

The last step here follows by elementary algebra. Now as E approaches 1 we see that S_E approaches $a^{n+1}/n + 1$. Thus we have

$$\int_0^a x^n \, dx = \frac{a^{n+1}}{n+1}.$$

This argument works for positive integers. Do you see where it fails when $n = 1/2$? Fermat was able to extend his idea to include all rational values of n except the logarithmic case, $n = -1$.

The above proof is quite easy for us to understand, but this is primarily because we have translated it into modern notation and nomenclature. Fermat used proportions in his argument so it is fairly difficult to understand. A translation error in [4] (line 21 on page 220 should be "increasing," not "decreasing") makes the original argument even harder.

EXERCISES

1. There are very few problems where unequal subdivisions are useful, but here is one. Use the definition of the derivative to show

$$\int_0^a \sqrt{x} \, dx = \frac{2}{3} a^{2/3}.$$

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Use the n partition points $x_k = bk^2/n^2$ and the right endpoints of these intervals as evaluation points. [From George F. Simmons, *Calculus with Analytic Geometry*, McGraw Hill, 1985, p. 176.]

REFERENCES

[1] Boyer, Carl B., "Fermat's integration of x^n," *National Mathematics Magazine*, 20(1945), 29–32.

[2] Mahoney, Michael S., *The Mathematical Career of Pierre de Fermat (1601–1665)*, Princeton University Press, 1973. See especially pp. 243–253.

[3] Mahoney, Michael S., "Fermat, Pierre de," *Dictionary of Scientific Biography*, 4, 566–576.

[4] Struik, Dirk J., "Fermat. Integration," in *A Source Book in Mathematics, 1200–1800*, Harvard University Press, 1969, pp. 219–222.