Schwarz's Paradox

After receiving his doctorate from the University of Turin in 1880, Giuseppe Peano (1858–1932) became the assistant (a new position at Turin designed to give the best students an entry into the academic world) to one of the mathematics professors. During the the 1881-1882 academic year, Peano was the assistant of Angelo Genocchi (1817–1889) who held the chair of infinitesimal calculus and who had been Peano's calculus teacher. Genocchi, who was 65 years old, had to ask Peano to teach his class because of his failing health. Calculus was a second year course, with four classes and two recitations per week. When the topic of surface area came up, Peano naturally turned to the commonly accepted definition of Joseph Alfred Serret (1819–1885) of the area of a surface S bounded by a curve C:

> area is the limit of the elementary areas of the inscribed polyhedral surfaces P bounded by a curve Γ as $P \longrightarrow S$ and $\Gamma \longrightarrow C$, where this limit exists and is independent of the particular sequence of inscribed polyhedral surfaces which is considered.[1]

This definition seems quite plausible especially considering that it is just a generalization of the definition of arc length. But Peano found a counterexample to show that it would not work. In a class lecture of May 22, 1882, Peano gave a corrected definition of surface area.[2] This was the first of a long sequence of discoveries that Peano made regarding elementary calculus.

Naturally, he was quick to tell his old teacher, but was surprised to learn that Genocchi was already aware of the difficulty. But Genocchi had not discovered it himself. On December 20, 1880, Hermann Amadeus Schwarz (1843-1921) had written Genocchi informing him of the difficulty in Serret's definition.[3] After Peano told Genocchi of his discovery, Genocchi wrote his friend Charles Hermite (1822–1901) of the independent discoveries of Schwarz and Peano because Hermite had used the incorrect definition of Serret in his lectures. Hermite wrote Schwarz for details so that he could revise his course. Schwarz wrote up the details of his discovery and Hermite quickly included them in the second edition of his mimeographed lecture notes (1883).[4] This did not please Schwarz for he did not feel right in publishing a result that had already appeared in print. The originnl note of Schwarz was not published until the second volume of his collected works appeared in 1890.[5] In the meantime, Peano published his work in 1890.[6] Although Peano's work was published first, it was clear to all concerned that Schwarz had priority for this near simultaneous discovery.[7]

In an amazing example of simultaneous discovery, both Schwarz and Peano found the same example to show that Serre's definition will not work. They considered an ordinary cylinder of

[1] J. A. Serret, *Cours de calcul différential et intégral*, 2 vols., first edition 1868, p. 296, second edition 1880, p. 293. Translated, with commentary by Axel Harnack, *Lehrbuch der Differential- und Integralrechnung*, 2 volumes, 1884-85. Translation from Lamberto Cesari, *Surface Area*, Princeton University Press, 1956, pp. 24–25. Pp. 24–26 of Cesari contain an excellent treatment of the Schwarz paradox.

[2] See p. 143 of his lithographed course notes.

[3] U. Cassina, "L'area di una superficie curva nel carteggio ineditio di Genocchi con Schwarz ed Hermite," *Rend. Ist. Lomb. Sci. Lett.*, (3) 83 (1950), 311-328.

[4] C. Hermite, *Cours professé à la Faculté de Sciences* Paris, first edition 1881-82, second edition 1883.

[5] H. A. Schwarz, "Sur une definition erronée de l'aire d'une surface courbe," *Gesammelte mathematische Abhandlungen*, II, pp. 309–311, 369–370.

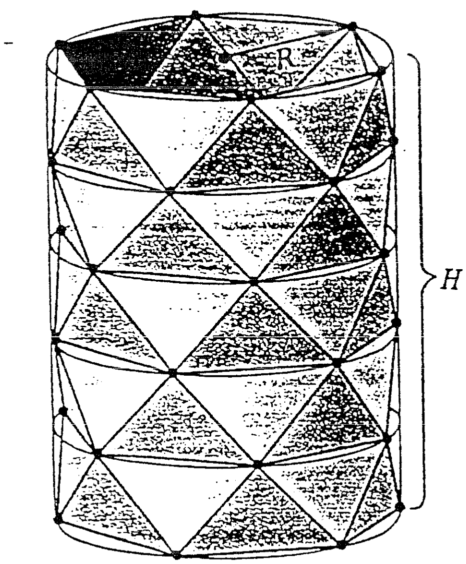
[6] G. Peano, "Sulla definizione dell'area d'una superficie," *Atti della Accademia Nazionale dei Lincei, Rendiconti, Classe di scienze fisiche, matematiche e naturali*, (4) 6 (1890), 54-57. English translation as "On the definition of the area of a surface (1890, 1903)," pp. 137–142 of *Selected Works of Giuseppe Peano*, translated and edited by Hubert C. Kennedy, University of Toronto Press, 1973. The '1903' refers to Peano's very interesting work, *Formulaire mathématique*, volume 4, where his example is given.

[7] Hubert C. Kennedy, *Peano. Life and Works of Giuseppe Peano*, Dordrecht: D. Reidel, 1980, pp. 7-10.

This file, Peano-area.tex, was typeset using TEX on May 30, 1996 at 1:22 A.M.

height H and radius R (Peano took $H = R = 1$.). Clearly the lateral surface area of this cylinder should be $2\pi RH$. What Schwarz and Peano did was to construct a sequence of inscribed polyhedral surfaces whose areas tend to any number not less than $2\pi RH$.

Divide the lateral area of the cylinder into n horizontal strips each of height H/n by using a sequence of parallel horizontal circles. Take m equally spaced points around each of these circles, but doing so in such a way that the points on each circle are midway between those on the circle above. By joining every pair of adjacent points on one circle to the point midway between on the circle below (and above), a polyhedral surface consisting of mn congruent isosceles triangles will be formed.

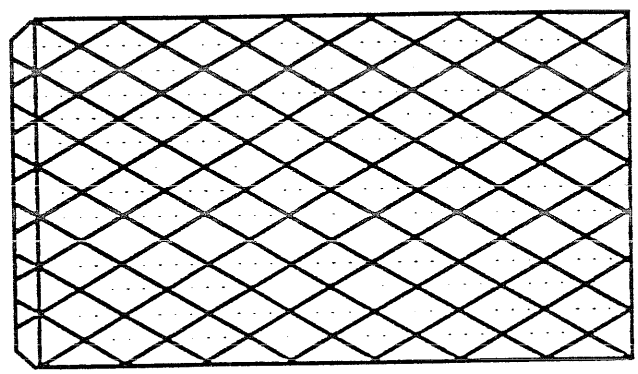


The sum of the areas are these mn triangles is

$$2mnR\sin\frac{\pi}{m}\sqrt{\frac{H^2}{n^2}+4R^2\sin^4\frac{\pi}{2m}}$$

To guarantee that the sides of all the triangles approach 0 take the limit as both m and n tend to infinity. If we let $m = n$ and then take the limit, we will get the desired area of $2\pi RH$. But if we first take $n = m^2$ we get $2\pi R\sqrt{1+R^2\pi^4/4}$. If first m tends to infinity and then n, the limit is infinity.[8]

To make a model of this polyhedral surface, take a piece of thin cardboard (a file folder works nicely) and mark it with a grid of diagonal lines as in the figure (the top of the figure is divided into m equal pieces, the side into n). Mark the back side of the cardboard with the dashed lines. Now lightly score each of the lines with a knife and fold back away from the lines. Next roll up the sheet into a cylinder, pressing to get the sheet to fold along the creases. Finally, glue the cylinder together using the flap.[9]



[8] These details are nicely worked out by Frieda Zames in "Surface area and the cylinder area paradox," *The Two-Year College Mathematics Journal*, 8 (1977), 207–211; reprinted in *A Century of Calculus, Part II, 1969–1991*, MAA, 1992, pp. 324–328 and also by Gail H. Atneosen, "The Schwarz paradox: An interesting problem for the first-year calculus student," Mathematics Teacher, vol. 65 (1972), pp. 281–284; reprinted in *Readings from the Mathematics Teacher. Calculus*, edited by Louise S. Grinstein and Brenda Michaels, NCTM, 1977, pp. 165–168.

[9] Dubrovsky, Vladimir, "In search of a definition of surface area. Now you see it, now you don't," *Quantum*, March/April 1991, pp. 6-9, and 44. The physical model of what they call Schwarz's boot, was designed by the Moscow architect and designer V. Gamayunov. It is described on p. 64.

This file, Peano-area.tex, was typeset using TEX on May 30, 1996 at 1:22 A.M.