Is it then possible to correlate the ρ variables $x_1, x_2, \ldots x_\rho$ with the one variable y so that to every determinate value-system $(x_1, x_2, \ldots x_\rho)$ there corresponds a determinate value y and conversely to every determinate value y there corresponds one and only one determinate value-system $(x_1, x_2, \ldots x_\rho)$?'

Although for years I held the opposite opinion to be true, it now seems to me that this question ought to be answered with *yes*, for the following reasons:

Every number $x \begin{smallmatrix}\geq 0\\ \leq 1\end{smallmatrix}$ can be expressed in one and only one way in the form of an *infinite* decimal fraction, so that:

$$x = \alpha_1 \cdot \frac{1}{10} + \alpha_2 \cdot \frac{1}{10^2} + \ldots + \alpha_\nu \cdot \frac{1}{10^\nu} + \ldots$$

where α_ν are integers ≥ 0 and ≤ 9. So every number x determines an infinite sequence $\alpha_1, \alpha_2, \ldots$ and conversely.

So we can write:

$$x_1 = \alpha_{1,1} \cdot \frac{1}{10} + \alpha_{1,2} \cdot \frac{1}{10^2} + \ldots + \alpha_{1,\nu} \cdot \frac{1}{10^\nu} + \ldots$$

$$x_2 = \alpha_{2,1} \cdot \frac{1}{10} + \alpha_{2,2} \cdot \frac{1}{10^2} + \ldots + \alpha_{2,\nu} \cdot \frac{1}{10^\nu} + \ldots$$

$$x_\rho = \alpha_{\rho,1} \cdot \frac{1}{10} + \alpha_{\rho,2} \cdot \frac{1}{10^2} + \ldots + \alpha_{\rho,\nu} \cdot \frac{1}{10^\nu} + \ldots$$

From these ρ numbers one can derive a $\rho + 1^{\text{st}}$ number y:

$$y = \beta_1 \cdot \frac{1}{10} + \beta_2 \cdot \frac{1}{10^2} + \ldots + \beta_\nu \cdot \frac{1}{10^\nu} + \ldots$$

if one takes:

(I) $$\beta_{(n-1)\rho+1} = \alpha_{1,n};\ \beta_{(n-1)\rho+2} = \alpha_{2,n};\ \ldots$$
$$\beta_{(n-1)\rho+\sigma} = \alpha_{\sigma,n};\ \ldots\ \beta_{(n-1)\rho+\rho} = \alpha_{\rho,n}$$

Since every positive integer ν can be expressed in one and only one way in the form:

$$\nu = (n-1)\rho + \sigma \quad \text{where} \quad \begin{matrix}\sigma > 0\\ \leq \rho\end{matrix},$$

one sees that the sequence $\beta_1, \beta_2, \ldots$ (and therefore also y) is completely determined by (I).

But also conversely, if one starts with the number y (and consequently with the sequence $\beta_1, \beta_2, \ldots$) then the ρ sequences:

$$\alpha_{1,1}, \alpha_{1,2}, \ldots$$
$$\ldots\ldots\ldots\ldots$$
$$\alpha_{\sigma,1}, \alpha_{\sigma,2}, \ldots$$
$$\ldots\ldots\ldots\ldots$$

$$\alpha_{\rho,1}, \alpha_{\rho,2}, \ldots$$

and consequently also the ρ numbers $x_1, x_2, \ldots, x_\rho$ are *uniquely* determined by the equations (I).

Dedekind to Cantor

The only objection that I can at the moment raise against your interesting theorem (which you will perhaps solve without difficulty) is the following. You say: 'Every number $x(\geq 0 \text{ and } \leq 1)$ can be expressed in one and only one way in the form of an *infinite* decimal fraction, so that:

$$x = \alpha_1 \cdot \frac{1}{10} + \alpha_2 \cdot \frac{1}{10^2} + \ldots + \alpha_\nu \cdot \frac{1}{10^\nu} + \ldots$$

where the α_ν are integers ≥ 0 and ≤ 9. So every number x determines an infinite sequence $\alpha_1, \alpha_2, \ldots$ and conversely.' The underlining of the word '*infinite*' makes me think that you exclude the case of an infinite fraction where an α_ν different from zero is followed only by the numerals $0 = \alpha_{\nu+1} = \alpha_{\nu+2} = \ldots$ and instead of writing

$$x = \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \ldots + \frac{\alpha_\nu}{10^\nu} + \frac{0}{10^{\nu+1}} + \frac{0}{10^{\nu+2}} + \ldots + \frac{0}{10^{\nu+\nu'}} + \ldots$$

always

$$x = \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \ldots + \frac{\alpha_\nu}{10^\nu} + \frac{9}{10^{\nu+1}} + \frac{9}{10^{\nu+2}} + \ldots + \frac{9}{10^{\nu+\nu'}} + \ldots$$

in order to exclude every possibility of a double representation of one and the same number. (The number $x = 0$ itself would have to be written in the form $0.0000\ldots$; but $x = \frac{3}{10}$ as $0.29999\ldots$.)

If this is your opinion (—one could of course equally well exclude the case that from a particular spot onwards only the numeral 9 appears; but then something similar would occur—) then my objection is the following. For the sake of simplicity, I restrict myself to the case $\rho = 2$, and set:

$$x = \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \ldots = 0.\alpha_1\alpha_2\ldots\alpha_\nu\ldots$$

$$y = \frac{\beta_1}{10} + \frac{\beta_2}{10^2} + \ldots = 0.\beta_1\beta_2\ldots\beta_\nu\ldots$$

and then like you form from the two numbers x, y the third number

$$z = 0.\gamma_1\gamma_2\gamma_3\ldots$$

where

$$\gamma_1 = \alpha_1,\ \gamma_2 = \beta_1,\ \gamma_3 = \alpha_2,\ \gamma_4 = \beta_2 \ldots \gamma_{2\nu-1} = \alpha_\nu,\ \gamma_{2\nu} = \beta_\nu, \ldots$$

Then z is a completely determinate function of the two continuous variables x, y and contained in the same interval $(0 \leq \tau \leq 1)$ But then there are infinitely many genuine fractions which are *never* equal to z, e.g.

$$0.4783105070 90\alpha_7 0\alpha_8 0\alpha_9 0 \ldots \alpha_\nu 0 \ldots$$

and by the same token every fraction $0.\gamma_1\gamma_2\gamma_3 \ldots$ in which from a definite point on either $\gamma_{2\nu-1}$ or $\gamma_{2\nu}$ is always equal to zero; for the converse derivation of x, y from such a z would lead to a non-existent (excluded) x or y.

I do not know if my objection goes to the essence of your idea, but I did not want to hold it back.

Brunswick, 22 June 1877

Cantor to Dedekind

[Postcard: Postmark 23.6.77]

Alas, you are entirely correct in your objection; but happily it concerns only the proof, not the content. For I proved *somewhat more* than I had realized, in that I bring a system $x_1, x_2, \ldots x_\rho$ of unrestricted real variables (that are ≥ 0 and ≤ 1) into one-to-one relationship with a variable y that does not assume all values of that interval, but rather all with the exception of certain y''. However, it assumes each of the corresponding values y' only *once*, and that seems to me to be the essential point. For now I can bring y' into a one-to-one relation with another quantity t that assumes all values ≥ 0 and ≤ 1.

I am delighted that you have found no other objections. I shall shortly write to you at greater length about this matter.

Halle, 25 June 1877

I sent you a postcard the day before yesterday, in which I acknowledged the gap you discovered in my proof, and at the same time remarked that I am able to fill it. But I cannot repress a certain regret that the subject demands more complicated treatment. However, this probably lies in the nature of the subject, and I must console myself; perhaps it will later turn out that the missing portion of that proof can be settled more simply than is at present in my power. But since I am at the moment concerned above all to persuade you of the correctness of my theorem, namely, the theorem:

(A) A continuous manifold extended in e dimensions can be correlated one-to-one with a continuous manifold of one dimension. Or (what is only another form of the same theorem): the points (elements) of a manifold extended in ρ dimensions can be determined by a real coordinate t in such a way that to every real value of t in the interval $(0 \ldots 1)$ there corresponds a point of the manifold, and conversely to every point of the M. there corresponds a definite value of t in the interval $(0 \ldots 1)$.

I allow myself to present another proof of it, which I found even earlier than the other.

I proceed from the theorem that every irrational number $e \gtrless_1^0$ can be expressed in a fully determinate manner in the form of an infinite continued fraction:

$$e = \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cdots + \cfrac{1}{\alpha_\nu + \vdots}}} = (\alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots)$$

where α_ν is a positive integer. To every irrational number $e \gtrless_1^0$ there corresponds a definite infinite sequence α_ν and conversely to every definite infinite sequence α_ν there corresponds a definite irrational number $e \gtrless_1^0$.

Now, if $e_1, e_2, \ldots e_\rho$ are ρ quantities independent of one another, of which each can assume all irrational numerical values of the interval $(0 \ldots 1)$ and only these values, then set:

$$e_1 = (\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,\nu}, \ldots)$$
$$e_2 = (\alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{2,\nu}, \ldots)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$e_\rho = (\alpha_{\rho,1}, \alpha_{\rho,2}, \ldots \alpha_{\rho,\nu}, \ldots);$$

and determine from them a $(\rho + 1)^{st}$ irrational number:

$$\partial = (\beta_1, \beta_2, \ldots, \beta_\nu, \ldots)$$

by the system of equations:

$$\text{(I)} \qquad \beta_{(n-1)\rho+1} = \alpha_{1,n}, \ldots, \beta_{(n-1)\rho+\sigma} = \alpha_{\sigma,n}, \ldots \beta_{n,\rho} = \alpha_{\rho,n}.$$

Then also conversely every irrational number $\partial \gtrless_1^0$ produces a definite system $e_1, e_2, \ldots e_\rho$ by means of (I).

It seems to me that here one does not run into the obstacle which you found in my earlier proof.

Now it is a matter of proving the following theorem:

(B) A variable number e which can assume all *irrational* numerical values of the interval $(0 \ldots 1)$ can be *one-to-one* correlated with a number x which takes on *all* values of this interval without exception.

For once this theorem (B) has been proven, one correlates one-to-one the variables previously designated by $e_1, e_2, \ldots e_\rho$ and ∂ to respective individual other variables

$$x_1, x_2, \ldots x_\rho, y$$

all of which have an unrestricted range in the interval (0 . . . 1). Then in this way too one establishes a one-to-one reciprocal relation of, on the one hand, the system

$$(x_1, x_2, \ldots x_\rho)$$

and, on the other, the single variable y; and this leads to the proof of theorem (A).

To demonstrate (B), one first brings all *rational* numbers of the interval (0 . . . 1) (including the end-points) into sequential form. They are:

$$r_1, r_2, \ldots, r_\nu, \ldots$$

The values that the variable e can assume are accordingly all values in (0 . . . 1) *with the exception of* the numbers r_ν.

In addition one takes some arbitrary infinite sequence ε_ν of *irrational* numbers in the interval (0 . . . 1), subject only to the conditions that $\varepsilon_\nu < \varepsilon_{\nu+1}$ and that $\lim(\varepsilon_\nu) = 1$ for $\nu = \infty$. Let f be a variable which can assume all real values $\substack{\geq 0 \\ \leq 1}$ except for the values ε_ν. Then the two limited variables e and f can be put into a reciprocal one-to-one relation to each other by the following definition:

If f is equal to no r_ν then the correspondence is:

$$e = f;$$

but if $f = r_\nu$, then the correspondence is $e = \varepsilon_\nu$; then one easily sees that also conversely: if e is equal to no ε_ν, then $f = e$ and if $e = \varepsilon_\nu$ then $f = r_\nu$.

The theorem (B) is now reduced to the following theorem:

(C) A number f that can assume all values of the interval (0 . . . 1) with the exception of certain ε_ν which satisfy the conditions: $\varepsilon_\nu < \varepsilon_{\nu+1}$ and $\lim \varepsilon_\nu = 1$ can be correlated one-to-one with a continuous variable x which takes on all values without exception of the interval (0 . . . 1).

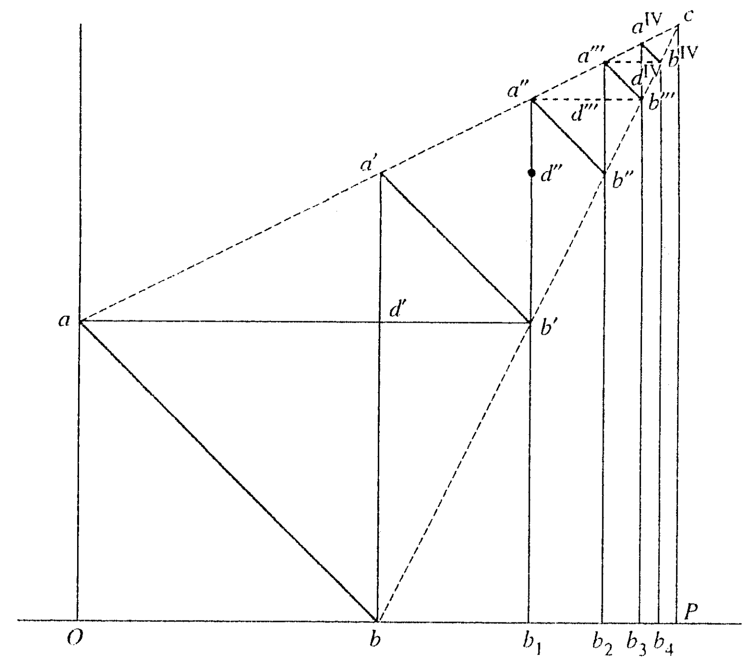
Here it is significant that the points $\varepsilon_1, \varepsilon_2, \ldots$ form a *series* and that the interval (0 . . . 1) is therefore decomposed by it into infinitely many sub-intervals.

And so, as you will not fail to see, this theorem (C) can be proved by successive application of the following theorem:

(D) A number y that can assume all values of the interval (0 . . . 1) with the solitary exception of the value 0 can be correlated one-to-one with a number x that takes on all values of the interval (0 . . . 1) without exception.

This theorem (D) can be seen to be true by considering the following peculiar curve.

My quantities x, y are the coordinates of a moving point m, and y is a single-



valued function of x; but while x assumes all values of the interval (0 . . . 1), the range of y is the same with *the solitary exception of the value 0.*

The curve consists of infinitely many parallel line segments $\overline{ab}$, $\overline{a'b'}$, $\overline{a''b''}$, and of the point c. *The end-points* $b, b', b'', \ldots$ *are not regarded as belonging to the curve.* The lengths are:

$$\overline{op} = \overline{pc} = 1;\ \overline{ob} = \frac{1}{2};\ \overline{bb_1} = \frac{1}{4},\ \overline{b_1b_2} = \frac{1}{8},\ \overline{b_2b_3} = \frac{1}{16}\ldots$$

$$\overline{oa} = \frac{1}{2};\ \overline{a'd'} = \frac{1}{4};\ \overline{a''d''} = \frac{1}{8};\ \overline{a'''d'''} = \frac{1}{16}, \ldots$$

For several years I have followed with interest the efforts that have been made, building on Gauss, Riemann, Helmholtz, and others, towards the clarification of all questions concerning the ultimate foundations of geometry. It struck me that all the important investigations in this field proceed from an unproven presupposition which does not appear to me self-evident, but rather to need a justification. I mean the presupposition that a ρ-fold extended continuous manifold needs ρ independent real coordinates for the determination of its elements, and that for a given manifold this number of coordinates can neither be increased nor diminished.

This presupposition had become my view as well, and I was almost convinced of its correctness. The only difference between my standpoint and all others was that I regarded that presupposition as a theorem which stood in great need of a proof; and I refined my standpoint into a question that I presented to several colleagues, in particular at the Gauss Jubilee in Göttingen. The question was the following:

'Can a continuous structure of ρ dimensions, where $\rho > 1$, be related one-to-one to a continuous structure of one dimension so that to each point of the former there corresponds one and only one point of the latter?'

Most of those to whom I presented this question were extremely puzzled that I should ask it, for it *is quite self-evident* that the determination of a point in an extension [Ausgedehntheit] of ρ dimensions always needs ρ independent coordinates. But whoever penetrated the sense of the question had to acknowledge that a proof was needed to show why the question should be answered with the 'self-evident' *no*. As I say, I myself was one of those who held it for the *most likely* that the question should be answered with a *no*—until quite recently I arrived by rather intricate trains of thought at the conviction that the answer to that question is an unqualified *yes*. Soon thereafter I found the proof which you see before you today.

So one sees what wonderful power lies in the ordinary real and irrational numbers, that one is able to use them to determine uniquely the elements of a ρ-fold extended continuous manifold *with a single coordinate*. I will only add at once that their power goes yet further, in that, as will not escape you, my proof can be extended without any great increase in difficulty to manifolds with an infinitely great dimension-number, provided that their infinitely-many dimensions have the form of a simple infinite sequence.

Now it seems to me that all philosophical or mathematical deductions that use that erroneous presupposition are inadmissible. Rather the difference that obtains between structures of *different* dimension-number must be sought in quite other terms than in the number of independent coordinates—the number that was hitherto held to be characteristic.

Halle, 29 June 1877

Please excuse my zeal for the subject if I make so many demands upon your kindness and patience: the communications which I lately sent you are even for me so unexpected, so new, that I can have no peace of mind until I obtain from you, honoured friend, a decision about their correctness. So long as you have not agreed with me, I can only say: *je le vois, mais je ne le crois pas*. And so I ask you to send me a postcard and let me know when you expect to have examined the matter, and whether I can count on an answer to my quite demanding request.

The proof of theorem (C) is considerably simplified by using the following symbolism:

If a, b are two variables that can be correlated one-to-one with each other, then one writes:

$$a \sim b.$$

And then, if $a \sim b$ and $b \sim c$, we also have:

$$a \sim c.$$

Moreover, if $a', a'', \ldots$ is a finite or infinite sequence of well-defined variables or constants which, taken pairwise, assume no common values, but, taken together, have a range [Spielraum] which is precisely that of the single variable a, then one sets:

$$a \equiv (a', a'', \ldots).$$

One then has the following theorem:

(E) If:

$$a \equiv (a', a'', \ldots)$$
$$b \equiv (b', b'', \ldots)$$

and moreover:

$$a' \sim b'$$
$$a'' \sim b''$$
$$a''' \sim b'''$$
$$\ldots\ldots\ldots\ldots$$

then:

$$a \sim b.$$

From (D) one obtains by the substitutions:

$$y = \frac{z-\alpha}{\beta-\alpha}; \; x = \frac{u-\alpha}{\beta-\alpha}$$

the following generalization of (D):

(F) A number z which can assume all values of an interval $(\alpha \ldots \beta)$ except α can be correlated one-to-one with a number u that takes on all values of the interval $(\alpha \ldots \beta)$ without exception.

From this we obtain the following theorem:

(G) A number w that takes on all values of the interval $(\alpha \ldots \beta)$ except for the two end-values α, β can be correlated one-to-one with a variable number u that assumes all values of the interval $(\alpha \ldots \beta)$.

Proof. Let γ be a number $\substack{<\alpha\\>\beta}$; w' a variable that assumes all values of the interval $(\alpha \ldots \gamma)$ with the exception of α and γ; w'' a variable that assumes all values of the interval $(\gamma \ldots \beta)$ with the exception of the one end-point β.

Then:

$$w \equiv (w', w'').$$

If now we designate by u'' a variable that assumes all values of the interval $(\gamma \ldots \beta)$ without exception, and by z a variable that assumes all values of the interval $(\alpha \ldots \beta)$ with the exception of α, then, by (F):

$$w'' \sim u'', \tag{2}$$

and so by (1) and (E):

$$w \sim (w', u'').$$

But $(w', u'') \equiv z$, so that:

$$w \sim z.$$

But by (F) one also has:

$$z \sim u, \text{ consequently also:}$$

$$w \sim u. \text{ QED.}$$

Now in order to prove (C) we decompose f into the variables $f', f'', \ldots$ and the isolated value 1, where f' assumes all values of the interval $(0 \ldots \varepsilon_1)$ with the exception of ε_1, $f^{(\nu)}$ all values of the interval $(\varepsilon_{\nu-1} \ldots \varepsilon_\nu)$ with the exception of the end-values $\varepsilon_{\nu-1}$ and ε_ν. Then one has:

$$f \equiv (f', f'', f''', \ldots f^{(\nu)}, \ldots, 1).$$

Now let x'' be a variable that assumes all values of $(\varepsilon_1 \ldots \varepsilon_2)$ without exception; let x^{IV} be a variable that assumes all values of $(\varepsilon_3 \ldots \varepsilon_4)$ without exception; let $x^{(2\nu)}$ be a var. that ass. all values of the int. $(\varepsilon_{2\nu-1} \ldots \varepsilon_{2\nu})$ without exception.
Then by (G):

$$f'' \sim x''$$
$$f^{IV} \sim x^{IV}$$
$$\ldots\ldots\ldots\ldots$$
$$f^{(2\nu)} \sim x^{(2\nu)}$$
$$\ldots\ldots\ldots\ldots$$

therefore:

$$f \sim (f', x'', f''', x^{IV}, \ldots f^{(2\nu-1)}, x^{(2\nu)}, \ldots, 1).$$

But one has:

$$(f', x'', f''', x^{IV}, \ldots f^{(2\nu-1)}, x^{(2\nu)}, \ldots, 1) \equiv x$$

and thus:

$$f \sim x.$$

Dedekind to Cantor

I have examined your proof once more, and I have discovered no gap in it; I am quite certain that your interesting theorem is correct, and I congratulate you on it. However, as I already indicated in the postcard, I should like to make a remark that counts *against* the conclusions concerning the concept of a manifold of ρ dimensions that you append in your letter of 25 June to the communication and the proof of the theorem. Your words make it appear—my interpretation may be incorrect—as though on the basis of your theorem you wish to cast doubt on the meaning [Bedeutung] or the importance of this concept; e.g. you say at the close of the letter, '*Now it seems to me* that all philosophical or mathematical deductions that use that erroneous presupposition [of the determinateness of the dimension-number] are inadmissible. Rather the difference that obtains between structures of *different* dimension-number must be sought in quite other terms than in the number of independent coordinates—the number that was hitherto held to be characteristic.'

Against this, I declare (despite your theorem, or rather in consequence of reflections which it stimulated) my conviction or my faith (I have not yet had time even to make an attempt at a proof) that the dimension-number of a continuous manifold remains its first and most important invariant, and I must defend all previous writers on this subject. To be sure, I gladly concede that the constancy of the dimension-number is thoroughly in need of proof, and so long as this proof has not been furnished one may doubt. But I do not doubt this constancy, although it appears to have been annihilated by your theorem. For all authors have clearly made the tacit, completely natural presupposition that in a new determination of the points of a continuous manifold by new coordinates, these coordinates should also (in general) be *continuous* functions of the old coordinates, so that whatever appears as continuously connected under the first set of coordinates remains continuously connected under the second. Now, for the time being I believe the following theorem: 'If it is possible to establish a reciprocal, one-to-one, and complete correspondence between the points of a continuous manifold A of *a* dimensions and the points of a continuous manifold B of *b* dimensions, then this *correspondence itself*, if *a* and *b* are *unequal*, is necessarily *utterly discontinuous*.' This theorem would also explain what happened with the first proof of your theorem, namely the incompleteness of the proof; the relation which you then wished to establish (by decimal fractions) between the points of a ρ-fold structure and the points of a unit interval would have been (if I do not deceive myself) *continuous*, if only it had also contained *all* points of the unit interval; similarly it seems to me that in your present proof the *initial* correspondence between the points of the ρ-interval (whose coordinates are all irrational) and the points of the unit interval (also with irrational coordinates) is, in a certain sense (smallness of the alteration) as continuous as possible; but to fill up the gaps, you are compelled to admit a frightful, dizzying discontinuity in the correspondence, which dissolves everything to atoms, so that every continuously connected part of the one domain appears in its image as thoroughly decomposed and discontinuous.

I hope I have expressed myself with sufficient clarity; the intent of my letter is only to ask you not to engage in public polemics against the article of faith that has hitherto been regarded as a fundamental truth of the theory of manifolds until you have thoroughly examined my objection.

Brunswick, 2 July 1877

Cantor to Dedekind

⟦Postcard: Postmark 2.7.77⟧

I am very happy that you have examined the proof and found it correct. I ask you to adhere to your plan, and to give me your views on the sense of the result more extensively and in greater depth; I should like to use them to form my judgement on the further pursuit of the subject.

Halle, 4 July 1877

I was very pleased by your letter of 2nd July, and thank you for your penetrating and exceptionally apt remarks.

In the conclusion of my letter of 25 June I unintentionally gave the appearance of wishing by my proof to oppose altogether the concept of a ρ-fold extended continuous manifold, whereas all my efforts have rather been intended to clarify it and to put it on the correct footing. When I said: 'Now it seems to me that all philosophical and mathem. deductions which use that erroneous presupposition—' I meant by this presupposition *not* 'the determinateness of the dimension-number' but rather the determinateness of the independent coordinates, whose number is assumed by certain authors to be in all circumstances equal to the number of dimensions. But if one takes the concept of coordinate *generally*, with no presuppositions about the nature of the intermediate functions, then the number of independent, one-to-one, complete coordinates, as I showed, can be set to any given number. I am also of your opinion that if we require that the correspondence be continuous, then only structures with the same number of dimensions can be related to each other one-to-one; and in this way we can find an invariant in the number of independent coordinates, which ought to lead to a definition of the dimension-number of a continuous structure.

However, I do not yet know how difficult this path (to the concept of dimension-number) will prove, because I do not know whether one is able to limit the concept of *continuous correspondence in general*. But everything in this direction seems to me to depend on the possibility of such a limiting ⟦Begrenzung⟧.

I believe I see a further difficulty in the fact that this path will probably fail if the structure ceases to be thoroughly continuous; but even in this case one wants to have something corresponding to the dimension-number—all the more so, given how difficult it appears to be to prove that the manifolds that occur in nature are thoroughly continuous.

By these lines, I wish only to indicate to you that, far from wishing to turn my result against the article of faith of the theory of manifolds, I rather wish to use it to secure its theorems, and, so far as possible, to make a contribution. For today I wish to take no more of your time, and I ask you, if you can find the time, to investigate the questions that press for an answer, not to despise them, and to let me know the results you obtain.

Halle, 23 Oct. 1877

. . . For a quarter year, Herr Borchardt has had a finished work of mine about the investigation I undertook last summer, entitled: *A contribution to the theory of manifolds* ⟦*Cantor 1878*⟧. I hope that it will appear shortly. Since I was able to use your friendly advice in writing it, perhaps you will be interested to know that I have found an even easier proof for one of my theorems. If two well-defined manifolds can be correlated to one another one-to-one and completely, element for element, then I say that they have the *same power* ⟦*Mächtigkeit*⟧ or that they are *equivalent*; similarly, I call two real variables *a* and *b equivalent* if they can be correlated to one another one-to-one and completely, and in this case I write, as you know,

$$a \sim b.$$

The theorem in question is as follows:

If e is a var. that assumes all irrational values $\substack{\geq 0 \\ \leq 1}$, and if x is a variable that takes on all rational and irrational values that are ≥ 0 and ≤ 1, then:

$$e \sim x.$$

Proof. Let ϕ_ν be the general term of a sequence that consists of all rational numbers ≥ 0 and ≤ 1; let η_ν be the gen. term of a sequence of any unequal irrational numbers $\substack{\geq 0 \\ < 1}$. E.g.:

$$\eta_\nu = \frac{\sqrt{2}}{2^\nu}$$

and let h denote the variable that takes on all values of the interval $(0 \ldots 1)$ with the exception of the ϕ_ν and the η_ν. Then:

$$(1) \qquad x \equiv \{h, \eta_\nu, \phi_\nu\}$$

$$e \equiv \{h, \eta_\nu\}.$$

For the last formula we can also write:

$$e \equiv \{h, \eta_{2\nu-1}, \eta_{2\nu}\}. \tag{2}$$

If one compares the formulae (1) and (2) and notes that:

$$h \sim h; \; \eta_\nu \sim \eta_{2\nu-1}; \; \phi_\nu \sim \eta_{2\nu}$$

then it follows that:

$$x \sim e, \text{ as was to be shown.}$$

Have you perhaps further investigated the question whether in the definition [Begriffsbestimmung] of an n-fold continuous manifold the condition of continuous correspondence suffices to make the concept safe against every contradiction and internally secure? . . .

P.S.—I have read some of Lipschitz's new *Textbook of analysis*; do you like it?

Dedekind to Cantor

1877.10.27

. . . However I still *believe* that the concept of dimension-number actually receives its invariant character from the condition of *continuous* correspondence.

The work of Lipschitz contains, so far as I have been able to see, much that is good and interesting; in accordance with my critical nature I have reservations about several points, but I find it very gratifying that he makes an earnest attempt to let mathematical rigour flourish even in a mathematical *textbook*. As for the justification of the theory of irrationals, in which he builds almost entirely upon your presentation (first published by Herr Heine), my greatest reservation is that (on p. 46) an entirely new *assumption* ('Then the boundary-value 𝔊 falls' etc.) is presented apparently as an obvious *consequence* of the earlier ones; moreover, I cannot grant the correctness of the final remark in §14 on the Greeks. . . .

Cantor to Dedekind

Halle a/Saale, 29th Dec. 1878

. . . You will probably also be in possession of the work, 'Fondamenti per la teorica delle funzioni di variabili reali di *Ulisse Dini*, Pisa 1878'. It seems to me to have been written with a deep knowledge of the subject and with great skill; he uses your method of introducing the numbers.—Although I agree with

it completely, still I believe that it is equivalent to the method I indicated in a paper on trigonometric series[d] and that by the formal distinction of numerical quantities of different order—by which I *only* wished to express the different ways in which they could be given by simple infinite sequences (whose terms approach infinitely close *to each other* as the index increases)—the danger does not arise that one could believe that I wished to extend the domain of the real numbers. I *never even remotely* intended such a blunder; I expressly say in my paper that every number I designate by c can be set equal to a number b. Moreover, this blunder has actually been made by somebody else, outrageous though this may sound. I do not know whether you know of Thomae's *Outline of a theory of complex functions and theta-functions*; in the second edition, p. 9, one finds numbers which (*horribile dictu*) are smaller than every thinkable real number and yet are different from zero.

Thomae, Lüroth, Jürgens, and, a few days ago in Borchardt's *Journal*, von Netto have written about the question whether continuous manifolds with different dimension-number can be correlated to each other one-to-one and continuously—or rather, about the theorem that says that this is not possible; but the matter does not seem to me to be fully resolved.

Halle a/Saale, 17 Jan. 79

I think I have settled the question about the one-to-one and *continuous* mapping [Abbildung] of manifolds that was suggested by my investigations, and done so in the simplest and most rigorous manner, by reducing the question to the familiar fundamental theorem of analysis, according to which:

(I) a continuous function of *one* continuous variable t which has a negative value for $t = t_0$ and a positive value for $t = t_1$ assumes the value zero at least once in between.

The attempts of Thomae and Netto suffer, as you will perhaps have seen, from imperfections; thus, e.g., Thomae relies on a theorem* of Riemann's (*Gesammelte Werke* p. 450: A manifold [Vielstreck] of less than $n - 1$ dimensions, etc., etc.) which is, *for him*, unprovable; while, as I have now clearly seen, this theorem is as it were equivalent to the theorem to be proved for the case $\nu = n - 1$; but since $n - 1$ is as general as the number n, Thomae's proof revolves in a circle.

The general proof, which I shall give in a moment, has in fact been known to me for some time (over a year); but I did not formerly think it was rigorous, and refrained from speaking of it. The discovery which I made several days ago

[d] [*Cantor 1872*.]
* He calls it an axiom.