I now have an obligation to investigate all the possible ways in which they can be applied to the structures of algebraic as well as transcendental geometry; it seems unlikely that the universality and sharpness of the results can be surpassed by other methods.

D. ON AN ELEMENTARY QUESTION IN THE THEORY OF MANIFOLDS (*CANTOR 1891*)

This article announces Cantor's 'diagonal argument' for proving the existence of non-denumerable sets, and, more generally, for proving that, for any set X, the cardinality of the power-set $\mathfrak{P}(X)$ is greater than the cardinality of X. The translation is by William Ewald; references to *Cantor 1891* should be to the paragraph numbers, which have been added in this edition.

⟦1⟧ In the article entitled: 'On a property of the set of all real algebraic numbers' (*Journ. Math.* Vol. 77, p. 258)[a] a proof is given, probably for the first time, of the theorem that there are infinite manifolds which cannot be correlated in a reciprocal one-to-one way with the totality ⟦Gesamtheit⟧ of all finite integers 1, 2, 3, ..., ν, ...; or, as I am accustomed to saying, which do not have the power of the number-sequence 1, 2, 3, ..., ν, ... That is, from the propositions proved in §2 it follows immediately that, for example, the totality of all real numbers of an arbitrary interval ($\alpha \ldots \beta$) *cannot* be presented in the sequential form

$$\omega_1, \omega_2, \ldots, \omega_\nu, \ldots$$

⟦2⟧ But it is possible to give a much simpler proof of that theorem which does not depend on considering the irrational numbers.

⟦3⟧ Namely, if m and w are any two mutually exclusive characters ⟦Charaktare⟧ we consider a set ⟦Inbegriff⟧ M of elements

$$E = (x_1, x_2, \ldots, x_\nu, \ldots)$$

which depend on infinitely many coordinates $x_1, x_2, \ldots, x_\nu, \ldots$ where each of these coordinates is either m or w. Let M be the totality of all elements E.

⟦4⟧ The following elements, for example, belong to M:

[a] ⟦*Cantor 1874*; translated above.⟧

$$E^{\mathrm{I}} = (m, m, m, m, \ldots),$$
$$E^{\mathrm{II}} = (w, w, w, w, \ldots),$$
$$E^{\mathrm{III}} = (m, w, m, w, \ldots).$$

I now maintain that such a manifold M does not have the power of the sequence 1, 2, ..., ν, ...

⟦5⟧ This follows from the following proposition:

'If $E_1, E_2, \ldots, E_\nu, \ldots$ is any simply infinite ⟦einfach unendliche⟧ sequence of elements of the manifold M, then there is always an element E_0 of M which corresponds to no E_ν.'

For proof, let

$$E_1 = (a_{1,1}, a_{1,2}, a_{1,\nu}, \ldots),$$
$$E_2 = (a_{2,1}, a_{2,2}, a_{2,\nu}, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$E_\mu = (a_{\mu,1}, a_{\mu,2}, a_{\mu,\nu}, \ldots).$$
$$\ldots\ldots\ldots\ldots\ldots$$

Here the $a_{\mu,\nu}$ are determinately m or w. We now define a sequence $b_1, b_2, b_\nu, \ldots$, such that b_ν is equal to m or to w but is different from $a_{\nu,\nu}$.

⟦6⟧ That is, if $a_{\nu,\nu} = m$, then $b_\nu = w$, and if $a_{\nu,\nu} = w$, then $b_\nu = m$.

⟦7⟧ If we then consider the element

$$E_0 = (b_1, b_2, b_3, \ldots)$$

of M one sees at once that the equation

$$E_0 = E_\mu$$

can be fulfilled by no integral value of μ, since otherwise for the μ in question and for all integral values of ν,

$$b_\nu = a_{\mu,\nu},$$

and so in particular we would have $b_\mu = a_{\mu,\mu}$, which is excluded by the definition of b_ν. From this proposition it follows immediately that the totality of elements of M cannot be brought into the sequential form:

$$E_1, E_2, \ldots, E_\nu, \ldots;$$

otherwise, we would have the contradiction that a thing ⟦Ding⟧ E_0 would be an element of M as well as not an element of M.

⟦8⟧ This proof is remarkable not only because of its great simplicity, but more importantly because the principle followed therein can be extended immediately to the general theorem that the powers of well-defined manifolds have no maximum, or, what is the same thing, that for any given manifold L

we can produce a manifold M whose power is greater than that of L.

[9] Let, for instance, L be a linear continuum, say the totality of all real numbers which are ≥ 0 and ≤ 1.

[10] Let M be the totality of all single-valued functions $f(x)$ which take only the values 0 or 1, while x runs through all real values which are ≥ 0 and ≤ 1.

[11] That M does *not* have a *smaller* power than L follows from the fact that subsets of M can be given which have the same power as L—for instance, the subset which consists of all functions of x which have the value 1 for a single value x_0 of x, and for all other values of x have the value 0.

[12] But M does *not* have *the same* power as L: for otherwise the manifold M could be brought into a reciprocal one-to-one correspondence with the variable z, and M could be thought of in the form of a single-valued function of the two variables x and z

$$\phi(x, z)$$

such that to every value of z there corresponds an element $f(x) = \phi(x, z)$ of M, and, conversely, to every element $f(x)$ of M there corresponds a single determinate value of z such that $f(x) = \phi(x, z)$. But this leads to a contradiction. For if one understands by $g(x)$ the single-valued function of x which takes on only the values 0 and 1 and is different from $\phi(x, x)$ for every value of x, then on the one hand $g(x)$ is an element of M, and on the other hand $g(x)$ cannot arise from any value $z = z_0$ of $\phi(x, z)$, because $\phi(z_0, z_0)$ is different from $g(z_0)$.

[13] But if the power of M is neither smaller than nor equal to that of L, it follows that it is greater than the power of L (see *Crelle's Journal*, Vol. 84, p. 242).[b]

[14] In the 'Foundations of a general theory of manifolds'[c] I have already shown, in an entirely different manner, that the powers have no maximum; there it was even proved that the totality [Inbegriff] of all powers, if we think of these as ordered according to their size, forms a 'well-ordered set', so that in Nature there is for every power a next greater, and moreover every infinite ascending set of powers is followed by a next-greater.

[15] The 'powers' represent the unique and necessary generalization of the finite 'cardinal numbers'. They are none other than the actual-infinite cardinal numbers, and they have the same reality and determinateness as the others, except that the lawlike relations among them—their 'number theory'—is in part of a different sort than in the domain of the finite.

[16] The further development of this field is a task for the future.

[b] [*Cantor 1878*; reprinted in *Cantor 1932*, pp. 119–33.]
[c] [*Cantor 1883d*; translated above.]

E. CANTOR'S LATE CORRESPONDENCE WITH DEDEKIND AND HILBERT

The letters that follow were written by Cantor in the late 1890s; they deal with his distinction between 'consistent' and 'inconsistent' multiplicities, and therefore with what later became known as the 'paradoxes of set theory'.

The central events leading to the discovery of the set-theoretic paradoxes are as follows.

In an article dated February 1897 Cesare Burali-Forti published a purported proof of the following theorem: 'There exist *transfinite ordinal numbers* (or *order types*) a and b such that a is not equal to b, not smaller than b, and not larger than b.' He did *not* claim to have found a paradox in Cantor's theory, but merely to have proved his theorem by a routine *reductio ad absurdum* argument. (Burali-Forti's paper (*1897a*) in fact contained a misreading of Cantor's definition of well-ordering, as Burali-Forti realized when he read Cantor's proof of the trichotomy theorem (*Cantor 1897*); he acknowledged the misreading in *Burali-Forti 1897b*. Both Burali-Forti papers are translated in *van Heijenoort 1967*.).

Cantor's letter to Hilbert of 26 September 1897 (translated below) contains a *reductio* argument that every power is an aleph; the argument shows that he was aware of the necessity for distinguishing between *transfinite* and *absolutely infinite* sets (which he was soon to call *consistent* and *inconsistent* multiplicities).[a] But it does not treat this problem as a paradox endangering his theory of sets.

By 16 April 1902 (at the latest) Ernst Zermelo had discovered a version of Russell's paradox; he stated it to Edmund Husserl in the form of a proof that a set which contains each of its subsets as elements is inconsistent. (The dated fragment, in Husserl's hand, containing Zermelo's proof was published by *Rang and Thomas 1981*.) In a footnote to his *1908a*, Zermelo says that he had discovered Russell's antinomy independently of Russell 'and had communicated it prior to 1903 to Professor Hilbert among others'.

In 1903, Russell published his paradox in *The principles of mathematics*; for the first time, the Burali-Forti problem and the Russell paradox were treated as paradoxes rather than as *reductio* arguments. (For a detailed account of the way in which Russell came to regard his paradox *as* a paradox, see *Moore and Garciadiego 1981*.)

[a] This letter seems to be the earliest mention of the paradoxes in Cantor's correspondence with Hilbert. Felix Bernstein says that Cantor discovered his 'contradiction' in 1895 and communicated it by letter to Hilbert in 1896 and to Dedekind in 1899. (*Bernstein 1905a*, p. 187). Others have repeated this assertion. The original published source (cited by Bernstein) appears to be *Jourdain 1904*. Jourdain in turn was relying on information supplied by Cantor, who wrote to Jourdain on 4 November 1903 that he had conveyed the proof of the aleph theorem to Hilbert 'about 7 years ago'; but the letter in fact stems from September 1897. (The Cantor–Jourdain correspondence is published in *Grattan-Guinness 1971*.) See also Bernstein's remarks translated above, p. 836.