The Impact of Ballistics on Mathematics

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October 17, 2008

1 Abstract

In the first half of the 18th century, Benjamin Robins, a British mathematician and military engineer, invented the ballistic pendulum. This device allowed for fairly accurate estimates of the muzzle velocities of muskets and other artillery. Through this experimental work he discovered that air resistance should not be neglected. In 1742 he published these results in *New Principles of Gunnery*, the first book to deal extensively with external ballistics. This book motivated a deeper analysis of projectile motion — a topic tackled by Leonhard Euler and Daniel Bernoulli. Subsequently, Frederick the Great encouraged Euler to translate this work of Robins. Euler, true to form, tripled the length of the work with his annotations and published them in 1745. The annotated text was translated back into English in 1777, which, two and a half centuries later, brings us to our theme here.

2 The Early Theory of Ballistics

Unbelievable as it may sound, Aristotle — who was a thinker, not an observer — claimed that when an object is thrown up, it moves in a straight line until it runs out of "up," and then falls down, straight down. Aristotle believed that an object moved when a force was applied to it and stopped when the force stopped. This caused problems when trying to explain why an arrow shot from a bow continues to fly after it leaves the bow. Did the moving arrow create a vacuum behind it into which air rushed and thereby applied a force?[1]

Niccolò Tartaglia (1500–1557), was the first to use mathematics to study of the paths of cannonballs. He published two books, *Nova scientia* (1537) and *Quesiti et Inventioni diverse* (1546), which deal with gunnery. A partial English translation of the latter work appeared in 1588 [43]. In contrast to Aristotle, he argued that the trajectory of a cannonball is curved near its apex. More interestingly, he stated that the maximum range of a projectile is attained when the firing elevation is 45°, however he gave only the weakest argument to support this claim.

In the second half of his greatest work, *The Dialogues of the Two New Sciences* (1638), Galileo took up the question of projectile motion. He argued convincingly that the path of a projectile in a vacuum was a parabola. This conclusion served gunners well — at least in certain circumstances.[2] Evangelista Torricelli wrote a book, *De motu* that greatly impressed Galileo. It contained a geometrical way of computing the range of a projectile.[3] Also important in our pre-history of scientific gunnery is Newton's *Principia* (1787) where he argued, among much else, that air resistance is proportional to the square of the speed of the projectile.[4]

3 The Life of Benjamin Robins (1707–1751)

Benjamin Robins is best known to mathematicians as one of the warriors in *The Analyst* controversy. But our interest in him is his role in the ballistics revolution.

Benjamin Robins was born in 1707 — the same year as Euler — in Bath, England. He was the only son of poor Quaker parents, John and Sarah (née Broughton). They did not particularly value education, yet young Benjamin soon was recognized as a mathematical prodigy with a talent for languages and literature. This led to his introduction to Dr. Henry Pemberton (1694-1771), the physician and mathematician who is best known as the editor of the third edition of Newton's *Principia* (1727).[5] Robins then moved to London where he studied the works of Apollonius, Archimedes, Fermat, Huygens, DeWitt, Sluse, James Gregory, Barrow, Newton, Taylor and Cotes. Dr. James Wilson, M.D., a faithful friend of

[1] Aristotle, *Physics*. Items which are only cited once in this paper are cited in footnotes. Other items are in §9, References.

[2] See http://www.mcm.edu/academic/galileo/ars/arshtml/arstoc.html by Joseph Dauben.

[3] Philip J. Robinson, "Evangelista Torricelli," *The Mathematical Gazette*, Vol. 78, No. 481 (March 1994), pp. 37–47. Swetz, Frank J., "An historical example of mathematical modeling: the trajectory of a cannonball," *International Journal of Mathematical Education in Science and Technology*, vol. 20, no. 5, 731–741. Reprinted in *Learn from the Masters*, MAA 1995.

[4] For a richly illustrated discussion of the early history of gunnery see *The Geometry of War* by Jim Bennett and Stephen Johnson, which is the catalog of an exhibition at the Museum of the History of Science at Oxford in 1996.

[5] While studying medicine at Lyden under Boerhaave, and later at Paris and London, Pemberton taught himself mathematics. He published a paper in the *Philosophical Transactions* in 1727 attacking the mechanics of Leibniz and declaring strong support for Newton. This brought an introduction to Newton and within the year he was asked to edit the third edition of Newton's *Principia*.

Robins and editor of his *Mathematical Tracts* (1761) has written a long preface to that work [32, pp. v–xlvi] that is the best biographical source about Robins.[6] Of these works that Robins read, Wilson says "These authors he readily understood without any assistance; of which he gave frequent proofs to his friends." [32, p. vi].

In the early 1730s, Robins was interested in fortification design, bridge design, hydraulics and ballistics. This led to numerous trips to the United Provinces, Flanders, and Northern France with high ranking friends in British Society [40, p. 69].

On the mathematical side, his "A demonstration of the 11th proposition of Sir Isaac Newton's Treatise on Quadratures," which was published in the *Philosophical Transactions* (1727) was sufficiently well regarded that he was elected a Fellow of the Royal Society of London in that same year.

Then Robins turned to polemical writing. The first work to provoke him was *The Analyst* (1734) of Bishop George Berkeley. To discuss this will take us too far afield.[7] In May 1728 in the journal *The Present State of the Republic of Letters*, Robins published "Remarks on a Treatise, lately printed at Paris, and entitled Discours sur les Loix de la communication du Mouvement, par Mons. Bernoulli," [32, reprinted, pp. 174–188]. In 1739 he published *Remarks on M. Euler's Treatise of Motion* which was a logical attack on Euler's use of infinitesimal methods and also of his fondness of algebraic over geometric methods [17]. Robins was a strong defender of Euclid and his methods of proofs, something that Wilson brings out in his preface to the *Mathematical Tracts* of Robins.

Also in 1739, Robins wrote three political pamphlets railing against Sir Robert Walpole's reluctance to go to war with Spain and the corruption of Walpole's administration. The positive effect of Robins's support for the Tory opposition was that he was appointed secretary of a committee set up by the House of Commons to investigate allegations against Walpole. This ended in a "compromise between the contending parties; and most concerned were gratified, either with honours or places, except the secretary; which some attributed to his having been too sincere in the affair." [32, p. xxvii].

At this time, Sir Robert Walpole was making plans to found the Royal Military Academy at Woolwich to provide mathematical and scientific education for British engineering and artillery officers. The Academy was officially founded by a Royal Warrant signed 30 April 1741 by King George the Second.

Benjamin Robins was much interested in becoming the mathematics professor at this new institution, so he began to plan a course of lectures on fortification and gunnery. He showed a manuscript of this course to British political and military authorities as part of

[6]Of Wilson we know very little. He attended Trinity College, Cambridge where he was made a scholar in 1709 and M.D. in 1728. He traveled to Paris and Leyden with Pemberton to study anatomy. His appendix to the *Mathematical Tracts* [32] shows "a high and surprising degree of intimacy with historical and then current mathematical topics" [19, p. 348].

[7]The editorial introduction by Douglas M. Jesseph to *De Motu and The Analyst: A Modern Edition, with Introductions and Commentary* provides an excellent discussion of Berkeley's critique of the calculus. See also Niccolò Guicciardini, *The Development of the Newtonian Calculus in Britain, 1700–1800* (1989), pp. 38–45. The Jurin-Robins dispute is well documented at http://www.maths.tcd.ie/pub/HistMath/People/Robins/RepublickOfLetters/

his application process. Unfortunately, this appointment was blocked by Sir Robert Walpole who still felt the impact of Robins's attack. The mathematics professorship went to Mr. Derham, who served 1741–1743. He was so unknown that even his first name is unknown.[8] The professorship of the artillery staff went to John Muller (1699–1784), several of whose works were purchased by Sylvanus Thayer for the USMA library. One of these is the six volume set *A System of Mathematics, Fortification and Artillery, ... For the Use of The Royal Academy of Artillery at Woolwich* (1756-1757).

Instruction at Woolwich, both theoretical and practical, began in the fall of 1741 with Muller and Derham as masters. The King's Warrent called for instruction "in the several parts of Mathematics necessary to qualify them [the cadets] for service of the Artillery, and the business of Engineers" and also called for "Rules, Orders, and Regulations" to be drawn up by the Master-General of Ordinance [13, pp. 1–2]. Details of the course in 1741 are specific about the mathematics curriculum:

> VII. That the second Master shall teach the Science of Arithmetic, together with the principles of Algebra and the Elements of Geometry, under the direction of the Chief Master.
>
> IX. That the chief Master shall further instruct the hearers in Trigonometry and the Elements of the Conick Sections.
>
> To which he shall add the Principles of Practical Geometry and Mechanics, applied to raising and transporting great Burthens [sic];
>
> With the Knowledge of Mensuration, and Levelling, and its Application to the bringing of water and the draining of Morasses;
>
> And lastly, shall teach Fortification in all its parts. [13, p. 264].

Details are also given for instruction in gunnery, but no more mathematics is noted. There is one glaring ommision in this curriculum — there is no calculus. The explanation is simple. Up to this time, the Gallilean theory of gunnery, with parabolic trajectories arising by assuming that air-resistance is negligible, could be given in purely geometric terms; there was no need for calculus.

Having been denied the professorship, Robins sought to publish his work immediately and so the *New Principles of Gunnery* appeared in 1742. In 1746, Robins was the first to receive the Copley Medal, the highest award given by the Royal Society of London.

[8] It is likely that he was one of the "several children" of Dr. William Derham, F.R.S. (1657–1735), who was the first to measure the speed of sound. The younger Derham died in 1743 [20, p. 468]. Later 19th-century mathematics professors at Woolwich were better known. They are T. Simpson (served 1743–1761), J. L. Cowley (1761–1773), C. Hutton (1764–1807), J. Bonnycastle (1807–1821), O. Gregory (1821–1838), S. H. Christie (1837–1865), J. Sylvester (1855 [sic] – 1870), M. W. Crofton (1870–1884), and H. Hart (1884–??) [13, p. 260].

4 Editions of Robins

The first edition of the *New Principles of Gunnery* of Benjamin Robins appeared in 1742 [30]. It was followed by Euler's German translation of 1745 [6]. Then there were French editions of Robins 1742 in 1751 [31] and 1771 [33]. We have not yet seen these last two, although we have reason to believe that there is considerable commentary in them. Robins died in 1751 and his *Mathematical Tracts* of 1761 were edited by James Wilson, containing another edition of his work. Euler's 1745 edition was translated back into English in 1777; however, very little of Robins's work was included, just the enunciations of the propositions on which Euler commented.

The history of the French editions is interesting, so we include it here: On 23 August 1774, the economist and politician Anne-Robert-Jacques Turgot (1727-1781) wrote to Louis XVI:

> The famous Leonhard Euler, one of the greatest mathematicians of Europe, has written two works which could be very useful to the schools of the Navy and the Artillery. One is a Treatise on the Construction and Manoeuver of Vessels; the other is a commentary on the principles of artillery of Robins ... I propose that your Majesty order these to be printed. [Quoted, in English, by Clifford Truesdell, *An Idiot's Fugitive Essays on Science* (1984), p. 337]

Turgot had reason to know about these books, for at the time he was Ministre de la Marine. He was sensitive to the impropriety of translating a work without the author's permission so urged the king to compensate Euler with 5,000 francs from the secret accounts of the Navy. This French translation of Euler's work appeared in 1783 [34].

Knowing of Napoleon's interest in mathematics,[9] it is not surprising that he read this work. His annotations have been preserved, but unfortunately, they only deal with the first part of the work [27]. It is likely that Euler's mathematics was too much for Napoleon.

Robins 1742 is a small work of 152 pages. It begins with a Preface of 57 pages. There is a great deal of information about fortification and powder which we shall ignore. But on p. xxxix he starts talking about exterior ballistics and continues till the end, p. lvii. In this section Robins mentions several authors (but we have been forced to conjecture the titles of the books he intended):

- Tartaglia, *Nova Scientia* (1537) *Questi et inventioni diversi* (1546)
- Busca, Gabriello, fl. 1580, *Instrvtione de bombardieri* (1545). This is the first book for the professional artilleryman [Spaulding1937, p. 98]
- Collado, *Prattica manuale dell'artiglieria* (1641)

[9]See, for example, Jacquelyn Maynard, "Napoleon's Waterloo wasn't mathematics," *Mathematics Teacher*, v82, n8, pp. 648–54, Nov 1989 and, Christoph J. Scriba, "Wie kommt "Napoleons Satz" zu seinem Namen?" *Historia Mathematica*, 8 (1981), no. 4, 458–459.

- Ufano, *Tratado dela Artilleria y uso della platicada por el capitan Diego Ufano en las Guerras de flandes* (1613)
- Simienowicz, *Artis magnae artilleriae* (1650)
- Galileo, *Discorsi e dimostrazioni matematiche intorno a due nuove scienze attenenti all meccanica ed ai movimenti locali* (1638).
- Galeus,
- Ulrick,
- Blondel, *Nouvelle maniere de fortifier les places* (1684), (an allusion to Aristotle),
- Bourne,
- Eldred, William *The gunner's glasse* (1646) (http://www.dimacleod.co.uk/history/glass.htm)
- Anderson, Robert. *Art of Gunnery*, London, 1674. See *The Biographical Dictionary of the Society for the Diffusion of Useful Knowledge*, p. 567 which is on the web. He was a weaver by trade.
- Halley,
- Newton, *Philosophia naturalis principia mathematica* (1687)
- la Hire (I am not sure whether it is the father or the son; the father's lovely Allegory of Geometry hangs in the Toledo Museum of Art).

This list is further evidence that Robins was widely read in mathematics, even if Euler will later point out works that he should have cited in his *New Principles of Gunnery*.

The Preface is followed by two chapters. The first (65 pages),"On the Force of Gunpowder," deals with interior ballistics, i.e., what happens inside the barrel of the gun or cannon. The most noteworthy contribution of this chapter is his description of his ballistic pendulum, a device which can be used to determine the muzzle speed of shot.[10] This was a tremendous advance. Previously, muzzle speed was computed from the range of the projectile (just as we do in calculus classes today), but under the assumption that there is no air resistance. In 1809, the president of the Russian Commission on Artillery, I. G. Gogel' wrote that this work, together with later work of D'Arcy and Hutton "'put all the discoveries of earlier artillerists in the shade" [21, p. 7].

The second chapter (30 pages) is entitled "Of the Resistance of the Air, and of the Track described by the Flight of Shot and Shells."[11] It is this chapter that is of interest to us, for it deals with exterior ballistics, the path of the projectile in the air.

[10] For the physics of the ballistic pendulum, see [1]

[11] Shot is a solid ball, of stone or iron, that is fired from a cannon. Shells are hollow balls that are filled with powder and thrown from a mortar and are designed to burst into fragments as they ricochet along the ground. Francis Scott Key knew we were winning when he noted the "bombs bursting in air."

5 Projectiles in a Vacuum

The second chapter begins by stating that “the greatest Part of Authors” have established that air resistance is in “the duplicate Proportion of its Velocity,” i.e., resistance is proportional to the square of the velocity. Unfortunately, he does not name any of these authors. He claims that this rule is “excessively erroneous,” and plans to show this in this work.

The chapter then consists of seven propositions, but we shall only discuss one of them. The fifth proposition in the second chapter of the *New Principles of Gunnery* is a rather unpromising result:

> PROP. V. When a Cannon-Ball of 24 lb. Weight, fired with a full Charge of Powder, first issues from the Piece, the Resistance of the Air on its Surface amounts to more than twenty times its Gravity. [30, p. 84]

Robins argues for this proposition using experimental evidence.

The real weight of this proposition — if you will excuse the pun — is in the scholium, an explanatory note amplifying the proposition. Robins indicates that he gave this proposition (and the previous one) so that he could refute the views of the “Theorists who have professedly written on the Subject of Gunnery” [30, p. 84]:

> since, as it is agreed on all Sides, that the Track of Projectiles would be a Parabola, if there was no Resistance, it has from hence been too rashly concluded, that the Interruption, which the ponderous Bodies of Shells and Bullets would receive from so rare a Medium as the Air, would be scarcely sensible, and consequently that their parabolic Flight would be hearby scarcely affected. [30, p. 85]

The proposition gives an example where the air resistance is certainly not negligible, but Robins is an experienced polemicist and skillful writer, so he wishes to give indisputable evidence that it is untenable to hold that air resistance is negligible. To begin this argument, he states seven “Postulates” that govern the motion of projectiles in a vacuum. He comments that

> the Demonstrations of which may be found in almost every Writer on the common Theory of falling Bodies. [30, p. 85]

Robins now enumerates these Postulates, most of which will be familiar to anyone who has studied projectile motion in a calculus or physics class. We will state one of the well known ones, and one which is not so well known:

> Post. 1. If the Resistance of the Air be so small, that the Motion of a projected Body be in a Curve of a Parabola, the the Axis of that Parabola will be perpendicular to the Horizon, and consequently the Part of the Curve, in which the Body ascends, will be equal and similar to that in which it descends. [30, p. 85]

> Post. 5. If the Velocity, with which the Body is projected, be known, then this greatest horizontal Range may be thus found: Compute, according to the common Theory of Gravity, what Space the projected Body ought to fall through to acquire the Velocity with which it is projected, then twice that Space will be the greatest horizontal Range, or the horizontal Range, when the Body is projected in an Angle of 45° with the Horizon. [30, p. 86][12]

Robins plans to use these Postulates in a clever way. They all follow from the assumption that there is no air resistance, an assumption that Robins wishes to show is wrong. While it would suffice to show that any one of these Postulates is incorrect under the assumption that there is air resistance, Robins opts for a much stronger argument. He will show that each one of the Postulates is incorrect if there is air resistance. We shall give but one example of his argument:

> Prop. VI. The Track described by the Flight of Shot or Shells is neither a parabola, nor nearly a Parabola, unless they are projected with small Velocities. [30, p. 87]

Robins justifies this Proposition with a great deal of empirical evidence, some the result of his own experimentation, some extracted from a wide variety of sources. Here is one stunning example that shows how far from parabolic the path of a projectile can be:

> a Musketball 3/4 of an Inch in Diameter, fired with half its Weight of Powder from a Piece 45 Inches long, moves with a Velocity of near 1700 Feet in 1″. Now if this Ball flew in the Curve of a Parabola, its horizontal range at 45° would be found, by the firth Postulate, to be about 17 Miles. Now all the practical Writers assure us, that this Range is really short of half a Mile. [30, p. 87][13]

The work has one more interesting proposition, namely that bullets "are also frequently driven to the Right or Left of that Direction [which they were shot] by the Action of some other Force." [30, p. 91]. Today we know this as the Magnus effect, which is named after the German physicist and chemist Heinrich Gustav Magnus (1802–1870), who made an experimental study of the aerodynamic forces on spinning spheres and cylinders in 1852. Before Robins, the effect had been mentioned by Newton in 1672 (apparently in regard to tennis balls).

In concluding our discussion of *New Principles of Gunnery*, we must remark that Robins is a very clear writer. He takes pains to give sound arguments — often multiple arguments — to make his points. There is no use of higher mathematics in the book, in fact, no mathematics beyond arithmetic — it could be understood by anyone with the willingness to read it. The use of experimental evidence to make his points is the novel feature of the work.

[12] This result is due to Tartaglia, who gives it for an arbitrary angle of projection; see Swetz in note 3.

[13] For a modern example, on the Apollo 14 mission which landed on the moon in 1971 the last thing Alan Shepard did before climbing back aboard was to hit a couple of golf balls. He shanked the first but "The next one I hit pretty flush. Here it would have gone 30 yards, but because there's no atmosphere there, it went about 200 yards." [*The New York Times*, October 14, 2008, page D2]

6 Euler's Mathematicization of Robins Work

Shortly after arriving in St. Petersburg in 1727, Euler wrote a short note, "Meditatio in experimenta explosione tormentorum nuper instituta," (Meditation on experiments made recently on the firing of cannon) [9]. This is paper E853 is the Eneström catalog of Euler's works.[14]

E853 has achieved lasting fame as a footnote in mathematical history by being the first printed source in which e is used to represent the base of the natural logarithm. Euler wrote this paper at the end of 1727 or at the beginning of 1728, when he was just 21 years old, describing seven experiments performed between August 21 and September 2, 1727.

Euler wrote to Frederick the Great [include date] asking to develop the work of Robins:

> Since this research can contribute a lot to the perfection of artillery, especially if one took the trouble to develop it better, and illuminate it fully, I judge that the public would be able to profit rather considerably ... [11, p. 309]; quoted from [42, p. 300].

Needless to say, Frederick agreed with this suggestion. Euler's translation, together with its copious annotations, expanded Robins's work from 152 pages to a work of triple the size.[15] The contrast with the original is amazing, for Euler takes full advantage of his knowledge of higher mathematics. In the preface, Euler argues strongly for the value of calculus:

> Some are of the opinion that fluxions are applicable only in such subtle speculations and can be of no practical use; or, at most, whatever conclusions can be obtained by them are owing to the well known lower parts of mathematics; but what has just been said of artillery is sufficient to remove this prejudice. It may be affirmed, that many things which depend on mathematics cannot be explained in all their circumstances without the help of fluxions, and even the sublime part of mathematics has met with difficulties that have not been fully mastered. [8, p. 3]

Also in the Preface, Euler makes some comments that seem wholly inconsistent with what we know about his generous personality and Christian charity. He writes that Robins is "unacquainted with several books on the theory of artillery" or else he wants to "exalt the merit of his own discoveries" [8, p. 11]. While this judgment is harsh, it is justified, for Euler mentions works of Huygens, Keile, Hermann, Taylor, de la Hire, Papin, Brachus, Johann Bernoulli and Daniel Bernoulli. Note that this list is disjoint from what Robins read early in his life.

[14]The Euler Archive, http://www.math.dartmouth.edu/ euler/, established and maintained by Dominic Klyve and Lee Stemkoski, then graduate students at Dartmouth College. It is the best place to find copies of Euler's works, including secondary references and translations when available.

[15]Ambiguity about the size is due to the edition. Euler's 1745 German edition contained 720 pages, but that includes the translation of the work of Robins as well as Euler's comments, and the pages are small. The English translation of this (1777) contains 423 pages. It contains just the statements of the propositions of Robins, not the text. The 1922 reprint of the German in Euler's *Opera omnia* contains 409 pages.

The "Zweyte Anmerkung" for Proposition V is of exceptional interest. Here Euler proves each of the seven Postulates that Robins states about the movement of projectiles in a vacuum. But, contrary to what had been done before, Euler makes good use of the calculus in deriving these Postulates. While we need to examine a number of books on gunnery that were published before Euler's work in 1745, we are fairly certain that this is the first time that anyone has used the calculus to explain the Gallilean theory of projectile motion without air resistance.

It takes Euler six pages to derive all seven of the Postulates of Robins, so we cannot give the details here. But we shall sketch the argument. In some ways, Euler's derivations are remarkably similar to what we would do in a calculus class today, but there are some significant differences. First we shall consider the similarities.

Euler sets up a coordinate system and then resolves the initial velocity into horizontal and vertical components (he does not use this vocabulary). Then he considers the velocity at an arbitrary point on the trajectory (we would start with the acceleration due to gravity, but he does not). When he integrates the horizontal component he ignores the constant of integration because he knows that, since there is no horizontal force, the position is simply time times velocity. For the vertical component he definitely considers the constant of integration. Of course, his results agree exactly with what we would do today.

It is interesting that Euler takes the sine of $90°$ to be 1. This is a change from the old way of doing trigonometry, but then this paper was written at the same time that Euler was writing his *Introductio in analysin infinitorum* (1748), the work that introduced trigonometry on the unit circle.

7 Projectiles with air resistance

In Proposition 6 Robins left no doubt about the significant effect that air resistance has on military projectiles. Although it would have been sufficient to show that any one of his seven postulates fails, he empirically demonstrated that *none* of the postulates hold for a projectile in air. Now it is Euler's turn to provide mathematical support. In his analysis Euler uses some fairly sophisticated and complicated approximation methods.

First Euler derives the equations of motion for horizontal shot, then for a vertical shot. Lastly he attempts to derive equations for a shot made at an oblique angle to the horizon. (See! Easy as 1–2–3! Well, if you are Euler it is.) These cases are treated in Annotations 1, 2, and 3. His derivations fill 29 pages in Euler's translation of Robins, so we can only sketch the high points.

Remark I

Suppose the ball is fired horizontally from E to F. If the distance EF is not too great, the ball will not descend far. Such a shot is called point-blank.[16] But the ball does fall

[16]The word is due to Tartaglia. He measured the elevation of a gun with a gunners quadrant. The long arm was laid in the cannon barrell. It was conneced to a shorter arm by a scale in the shape of a quarter

somewhat, say to G, a point directly below F. The angle $\angle FEG$ is very small. Consider an arbitrary point P on EF and a point directly below, M, to which the ball will fall as it moves from E toward P. Now we shall let Euler speak for himself:

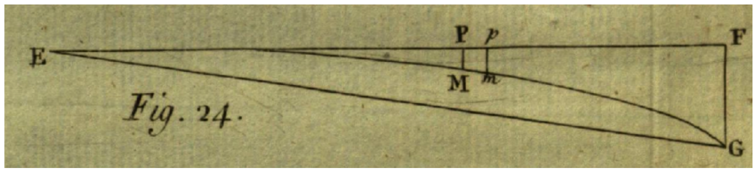

Figure 1: A point-blank shot

> "the line PM will be described by the action of gravity, which amounts to 15.625 Rhynland feet in a second. Let b be the height from whence a body must fall to acquire the velocity, which the body is projected with from E; let moreover the diameter of the ball $= c$, and the matter of which the ball consists n times heavier than air, let the shot in the time t advance to M or P, and let $EP = x$, $PM = y$, and the velocity of the ball $= \sqrt{v}$, since now $PM = y$ is the space through which a body will fall in the time t, so shall $y = t^2/4$, and to determine the motion in the horizontal line EP, it is to be observed, that the resistance in P is expressed by a column of air, whose height is
>
> $$\frac{v}{2} + \frac{v^2}{2h},$$
>
> [8, p. 278]

where the parameter v is the square of the velocity at P (or M), and the parameter h — which is determined empirically — is related to the "elasticity" of the air — its ability to fill in the vacuity formed behind the projectile as it moves.

The weight of the ball is expressed by a column of air whose height is $\frac{2}{3}nc$ where n is the ratio of the density of the ball to the density of air and c is the diameter of the ball. So the ratio of the resisting force to the weight of the ball is given by

$$\frac{3v(h+v)}{4nch} : 1.$$

This is the force that resists the motion of the ball, so Euler has the differential equations that govern the motion of a horizontal shot:

$$dv = \frac{-3v(h+v)}{4nch}\,dx \quad \text{and} \quad dt = \frac{dx}{\sqrt{v}}.$$

circle which was marked off with 12 points. To fire at 6 points meant to fire at 45°. To fire horizontally was point-blank, no points [43].

In typical Eulerian style, he makes three attempts to solve these equations. In the first, he attempts to solve these DEs without any simplifying assumptions. He is able to solve for x and t, but not v. In performing the integrations, his tools are limited to partial fractions and integration by substitution. The solutions he obtains are quite complicated:

$$x = \frac{4nc}{3}\, l\, \frac{b(h+v)}{v(b+h)} \quad \text{and} \quad t = \frac{8nc}{3}\Big(\frac{\sqrt{b}-\sqrt{v}}{\sqrt{bv}} - \frac{1}{\sqrt{h}}\,\text{A.tang.}\,\frac{(\sqrt{b}-\sqrt{v})\sqrt{h}}{h+\sqrt{bv}}\Big).$$

The l in the expression for x represents the hyperbolical or natural logarithm. In the expression for t, he notes that the arctangent is taken with respect to a circle of radius 1. That he feels obligated to make these remarks is indication that the notions have not yet become commonly accepted in the mathematical community.

Realizing that this solution is not as satisfactory as he wishes, he introduces a simplifying assumption, namely that

$$\exp\left(\frac{3x}{4nc}\right) = 1 + \frac{3x}{4nc}$$

Keeping only two terms of the exponential series quickly gives a solution: $x = \sqrt{b}x$. But this cannot be correct for this solution represents uniform motion. This is a nice example of Euler showing us his path to discovery. Even if it does not work, we can learn something from it. Need to cite Jerry Alexanderson's paper about Euler in Math Mag as he makes a comment about Euler showing his tracks.

In his third, and final attempt to provide a satisfactory solution, Euler takes three terms of the exponential series:

$$exp\Big(\frac{3x}{4nc}\Big) = 1 + \frac{3x}{4nc} + \frac{1}{2}\Big(\frac{3x}{4nc}\Big)^2.$$

This time he is able to solve for the time t to reach the point P, the angle through which the shot has fallen, and the velocity of the shot at P:

$$t = \frac{x}{\sqrt{b}} + \frac{3(b+h)xx}{16nch\sqrt{b}} + \frac{3(hh-bb)x^3}{128nncchh\sqrt{b}}.$$

$$\text{Angle}PEM = A.tang.\Big(\frac{x}{4b} + \frac{3(b+h)xx}{32ncbh}\Big).$$

and

$$v = b - \frac{3b(b+h)x}{4nch} + \frac{9b(b+h)(2b+h)xx}{32n^2c^2hh}.$$

Curiously, Euler makes no comments about the connections between these three attempts to solve the differential equations.

He concludes his first annotation with some numerical examples.

Remark II

Euler's interest in ballistics began during 1727, the first year that he was at the St. Petersburg Academy of Sciences. At the age of 20, Euler was present when, under the direction of General Gunther, the Master General of Ordnance in the Russian artillery corps, Daniel Bernoulli conducted several experiments involving cannon. Bernoulli, then age 27, published his results in the second volume of the Saint Petersburg *Commentarii*[17] and then described them again in his *Hydrodynamica* (1738)[18] his famous book on fluid dynamics, his most important work. He fired a small artillery piece vertically, timing the duration of the flight of the shot. More importantly, using Newton's quadratic law of air resistance, Bernoulli solved the differential equations of motion to compute the muzzle speed of the artillery piece [42, p. 284].

In this annotation, Euler considers a shot that is fired vertically. The mathematics here is similar to that in Remark III so we move immediately to it.

Remark III

Finally, Euler is ready for the general case, where the shot is fired at an oblique angle to the horizon.

The Track described by the Flight of Shot or Shells is neither a Parabola, nor nearly a Parabola, unless they are projected with small Velocities.

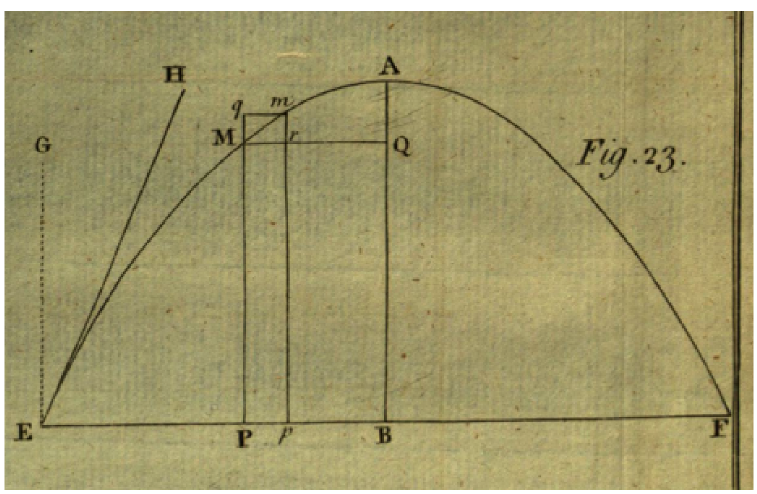

Figure 2: Shot at an angle

[17] "Dissertatio de actione fludiorum in corpora solida et motu solidorum in fludis," *Comentarii Academemiae Scientiarum Imperialis Petropolitanae*, II, 1727 (1729), 304–342 and "Dissertationis de actione fluidorum in corpora solida et motu solidorum in fluidis continuatio," *Commentarii*, III, 1728 (1732), 214–229. Euler only cites the first part of the paper.

[18] There is an English translation by Thomas Carmody and Helmut Kobus, Dover Publications, 1968.

In this section Euler endeavors to determine, or at least closely approximate, the path of a projectile that is shot at an angle. We assume that the projectile is spherical with diameter c, that its trajectory lies in a plane, and that the only forces acting on it are the force due to gravity and a resistant force due to the air. We also assume that this resistant force acts in a direction parallel and opposite to the direction of travel.

We consider the projectile at point M on its trajectory (see Figure 2) and its motion to point m over an infinitesimally small time interval dt. Thus its path from M to m may be assumed to be a straight line. The ball's velocity at M is denoted by $\sqrt{v}$ where v represents the height from which the body must fall to attain its velocity at M.[19] This velocity may be decomposed into its horizontal and vertical components.

As the ball moves from M to m the velocity will be diminished by both gravity and the resistant force due to air. If we let n denote the ratio of the density of the ball to the density of air, then, as Euler has previously discussed (Prop.II, Remark II), the weight of the body in air will be diminished by the weight of a quantity of air with the same volume as the body which displaces it. Thus, the vertical component of the velocity will be diminished by this gravitational force by $(1-\frac{1}{n})$. Euler denotes[20] the term $1-\frac{1}{n}$ by g.

Euler has also previously established (Prop. I & Prop. III, Note II) that the resistant force on a spherical projectile of diameter c is given by $\frac{3v(h+v)}{4nch}$ where h is the height of a column of air whose pressure is equal to that of the atmosphere. Thus the vertical component of the velocity will be diminished further by the vertical component of this resistant force, $\frac{3v(h+v)}{4nch}\sin\phi$ and the horizontal component of the velocity will be diminished by the horizontal component of the resistant force $\frac{3v(h+v)}{4nch}\sin\phi$, where $\phi = \angle mMr$ in Figure 2.

Thus, we arrive at the following system of differential equations

$$d[v(\sin\phi)^2] = -g\,dy - \frac{3v(h+v)\sin\phi}{4nch}dy, \tag{1}$$

$$d[v(\cos\phi)^2] = -\frac{3v(h+v)\cos\phi}{4nch}dx. \tag{2}$$

Find the differentials of the left-hand sides of (1) and (2) and substitute $dx = ds\cos\phi$ and $dy = ds\sin\phi$ into the right-hand sides to get

$$(\sin\phi)^2\,dv + 2v\sin\phi\cos\phi\,d\phi = \left(-g\sin\phi - \frac{3v(h+v)(\sin\phi)^2}{4nch}\right)ds, \tag{3}$$

$$(\cos\phi)^2\,dv - 2v\sin\phi\cos\phi\,d\phi = -\frac{3v(h+v)(\cos\phi)^2}{4nch}\,ds. \tag{4}$$

After some substitutions and algebraic manipulations with equations (3) and (4) we arrive at the differential equation

$$dv = -g\,dy - \frac{3v(h+v)}{4nch}ds. \tag{5}$$

[19]This strange way of expressing velocity goes back to Galileo. The problem was that it was difficult to measure short time intervals and hence velocity. See [28]

[20]Not to say that gravity is *equal* to this, but perhaps to indicate that this factor arises due to gravity. Its value will be determined empirically.

Euler now lets $dy = p\,dx$ and shows that

$$v = \frac{-g(1+p^2)dx}{2\,dp}. \tag{6}$$

Making the substitution $dp = q\,dx$, it then follows from (6) that

$$dv = -g\,dy + \frac{g(1+p^2)}{2q^2}dq. \tag{7}$$

Comparing (7) to (5) we see that

$$\frac{g(1+p^2)}{2q^2}dq = -\frac{3v(h+v)}{4nch}ds$$

We can now show that

$$\frac{4nc}{3}dq = (\sqrt{1+p^2}\,dp - \frac{g(1+p^2)\sqrt{1+p^2}}{2qh}\,dp. \tag{8}$$

Now, if we could write q as a function of p, then if follows from $dx = \frac{dp}{q}$ and $dy = \frac{p \cdot dp}{q}$ that

$$x = \int \frac{1}{q}dp \quad \text{and} \quad y = \int \frac{p}{q}dp. \tag{9}$$

In order to determine our constants of integration we can use the initial values given at point E, the beginning of the motion:

1. $x = 0$;
2. $y = 0$;
3. $p = dy/dx = \tan\theta$;
4. $q = \frac{-g(1+p^2)}{2b} = \frac{-g}{2b(\cos\theta)^2}$.

Euler notes that the integrals in (9) cannot be found in closed form, so he will attack with approximation methods. He has an arsenal of substitution and series techniques up his sleeves, and before he's done Euler will demonstrated his facility with a significant number of them. One would be hard pressed to call the mathematics that follows elegant. (Perhaps this is what Hardy was thinking of when he described ballistics as "repulsively ugly"?) It is clear that Euler is trying every trick he knows to force a useful solution to the problem.

In his first attempt at a solution Euler supposes that q can be written as a function of p and uses the substitutions $u = p/\sqrt{1+p^2}$ and $q = 1/r$. In this case it follows that r can be written as a function of u, and can thus be given in the form of a power series

$$r = a + Au + Bu^2 + Cu^3 + \ldots \tag{10}$$

Now, equation (8) can be rewritten to obtain

$$0 = k(1-u^2)^3 dr + r^2(1-u^2)\, du - \frac{r^3}{f}\, du \tag{11}$$

where $k = 4nc/3$ and $f = 2h/g$. We can replace r and dr in (11) with their respective series expansions given by (10) to obtain a differential equation in u. By using the initial value for u, i.e., $u(0) = \sin\theta$, we can solve for $A, B, C, \ldots$ in terms of a, k, and f. In particular,

$$A = \frac{a^2(a-f)}{kf}, \tag{12}$$

$$B = \frac{a^3(3a-2f)(a-f)}{2k^2f^2}, \tag{13}$$

$$C = \frac{a^2(3a-2f)}{3kf} + \frac{a^4(a-f)(15a^2-20af+6f^2)}{6k^3f^3}. \tag{14}$$

Using our substitutions for p and q and series (10), the integrals (9) can now be written as

$$x = \int \frac{1}{(1-u^2)^{3/2}} \left(a + Au + Bu^2 + Cu^3 + \ldots\right) du \tag{15}$$

and

$$y = \int \frac{u}{(1-u^2)^2} \left(a + Au + Bu^2 + Cu^3 + \ldots\right) du. \tag{16}$$

Thus, integrating term by term using known formulae and using the given initial values, we can give series formulations for x and y in u where the only coefficient to be determined is a, as we have already demonstrated how all other coefficients can be solved in terms of a. Moreover, since $u(t) = \sin\phi$, we can rewrite these series expansions for x and y in terms of trigonometric functions involving ϕ. Finally, the parameter a can be estimated from the equation:

$$-\frac{2b}{g}\cos^2\theta = a + A\sin\theta + B\sin^2\theta + C\sin^3\theta + \ldots \tag{17}$$

But, Euler is not happy with this result. He states

> As k increases, A, B and C diminish; but we see by this method of approximating, C cannot be less than A; for which reason we must try to get an approximation which will answer better.

So, Euler is going to attack again, this time starting with (11) written explicitly in terms of ϕ. If we substitute $u = \sin\phi$ and $du = \cos\phi\, d\phi$ into (11) and simplify we obtain

$$k\cos^5\phi dr + r^2\cos^2\phi\, d\phi = \frac{r^3}{f}\, d\phi \tag{18}$$

from which it follows that

$$x = \int \frac{r}{\cos^2\phi} d\phi = \frac{r\sin\phi}{\cos\phi} - \int \frac{\sin\phi}{\cos\phi} dr \tag{19}$$

and

$$y = \int \frac{r \sin\phi}{\cos^2\phi} d\phi = \frac{r}{2\cos^2\phi} - \frac{1}{2}\int \frac{1}{\cos^2\phi} dr \tag{20}$$

where $r = 1 + P + Q + ...$ After several integrations and some manipulation we obtain expressions for P and Q in terms of the variable ϕ and the parameter a, as well as an indication of how to find further terms if desired.

At this point Euler faces some very complicated equations. It appears that he is defeated for the time being, at least if he intends to find any sort of applicable results. But he will not completely surrender and concludes this "Remark" with a final endeavor to salvage some sort of useful result and arrive at some sort of final conclusion.

Euler notes that if there is no resistance, then we would have $r = a$ in which case the curve would be a parabola. Thus, if the resistance is small we could estimate r by $r = a + P$. In this case Euler is able to obtain some rather complicated expressions for x and y in terms of ϕ, as well as an estimate for a. Euler suggests that the expressions thus obtained for x and y might be used to estimate the range of such a projectile. He tells us that in order to find the range we need put $y = 0$ and find the angle of the trajectory at impact. However, he then asserts, "The equation which determines this angle is extremely complicated; and, indeed, the angle cannot be found otherwise than by a very prolix approximation." (Indeed, if Euler thinks that the approximation method to be used would involve an inordinate amount of effort, then we mere mortals should despair.) But, if we *could* find a value for this angle, then we can substitute it into the equation for x and thereby determine the range EF.

However, all is not lost! Euler tells us that we can find an equation expressing the relation between x and y via another approximation and our mathemagician presents us with the equation

$$\begin{aligned} y &= x\tan\theta - \frac{gx^2}{4b\cos^2\theta} - \frac{gx^3}{12bk\cos^3\theta} + \frac{g^2x^4\sin\theta}{96b^2k\cos^4\theta} + ... \\ &\quad - \frac{x^3}{6fk\cos^3\theta} + \frac{gx^4\sin\theta}{16fk\cos^4\theta} - \frac{gx^4}{48bk^2\cos^4\theta} - \frac{x^4}{24fk^2\cos^4\theta} + ... \end{aligned}$$

We are told that if the resistance be very small then this equation will very nearly describe the trajectory. Moreover, the range EF may also be estimated to be about

$$2b\sin 2\theta\left(1 - \frac{b(b+h)\sin\theta}{nch}\right)$$

given that $g \approx 1$ in this case. Note that the range in the case with no resistance would have been $EF = 2b\sin 2\theta$. So, Euler's final conclusions for the time being are that if the resistance is small, then

1. The range at any given angle of elevation will be less than that of the case with no air resistance;

2. As we increase the angle of elevation, the difference between the actual range and the range in the non-resistant case will increase;

3. The greatest range will occur at some angle less than 45°.

Moreover, if nc is much greater than b, i.e., we have a very large and/or very heavy shot moving at a very slow speed, we can estimate the angle that yields maximum range by

$$\sin\theta = \frac{1}{\sqrt{2}} - \frac{b(b+h)}{8nch}.$$

Unfortunately, this estimate is not useful in the case under consideration in Robins's Proposition VI and improvements must wait for a later date.

We note that in his later work Euler does eventually extract a useful method for obtaining fairly accurate projections for the ranges of various shot. This method is used to create detailed artillery tables, and hence we observe some of the benefits of using calculus in ballistics.

If you found these deductions difficult to follow, you are in good company. After his *Neue Grundsätze der Artillerie* was published in 1745, Euler sent a copy to Johann Bernoulli, his teacher and grand old man of mathematics. Bernoulli responded in a letter of 23 September 1745:

> I have recently received two treatises that you kindly sent me, one on cannons and the force of powder, the other containing the theory of the motion of planets and comets. For this double gift I am deeply grateful. I have now nearly finished reading the first of these books, but I have relied completely on the correctness of your computations and haven't verified them myself, for many of them seem too complicated to me.[21]

8 Impact

8.1 The Impact on Ballistics:

After Euler, there were two tracks in ballistic research, one theoretical, one experimental. In the 18th-century, important experimental work was done by D'Arcy in France and Hutton in Britain. Theoretical work was done by Borda, Lambert, and Legendre. The discussion of these developments must await further study.

For the modern theory of ballistics, see the books by Bliss (1944) [3], McShane, Kelley and Reno (1953) [25], and especially the modern bible of McCoy (1999) [24].

8.2 The Impact on Mathematics:

In the *Neue Grundsätze der Artillerie* "Euler's primary agendum was to explicate the power of the differential and integral calculus in analyzing practical engineering problems." [42, p. 290]. In this he succeeded admirably.

[21] P. Fuss, *Corréspondance mathématique et physique*, vol. 3, p. 597. Quoted by Mandryka [21, p. 15]

The Euler-Robins work was instrumental in influencing a shift from thinking about ballistics problems empirically to thinking about them analytically. Prior to their work, the general shape of the trajectory was important only in as far as it assisted in interpolating ranges from known experimental data. Thus, it didn't really matter that it was not accurate, because it was good enough. But with advances in fluid dynamics along with Euler's mathematization of the situation, here finally is a problem that mathematicians and scientists can sink their teeth into, eventually leading to the Navier-Stokes equations and the still open problem (of the century) of their solution in more than just a few special cases. Only after their work appears do we begin to see useful applications of theory in ballistics.

8.3 The Impact on Teaching:

We have noted in §3 the Woolwich curriculum in 1741 — before Euler's *Neue grundsätze der artillerie* appeared in 1745. To show how this impacted the teaching of mathematics, we give the Woolwich mathematics curriculum in 1772:

- The Elements of Euclid
- Trigonometry applied to Fortification, and the Mensuration of Superficies and Solids
- Conic Sections. Mechanics applied to the raising and transporting heavy bodies, together with the use of the lever pulley, wheel, wedge and screw, &c.
- The Laws of Motion and Resistance, Projectiles, and Fluxions.

One notes a significant change from the 1741 curriculum: Fluxions — the calculus — is taught to all students. This is a direct result of the work of Robins and Euler on gunnery. Artillery schools across Europe saw the necessity of teaching calculus. Such schools included Piedmont-Savoy which began teaching calculus in the 1750s, the Royal Artillery and Military Academy in Turin where Lagrange taught, the Prussian artillery corps where Jacobi and Templehof taught, the French regimental school at Auxonne, and the Australian Artillery Academy, to give a few examples [40] [42].

Calculus was taught at the École Polytechnique from its inception in 1794. Sadly, the same cannot be said of the United States Military Academy. Alden Partridge tutored a few cadets in calculus about 1807 and Andrew Ellicott taught a class calculus in 1813, but calculus was not taught to all cadets until the mid 1820s. The reason for this was not that the faculty did not see the need to teach calculus to future artillery officers, but that the incoming cadets were so weak in mathematics that there was no possibility of teaching calculus to all of them in the 1810s.

> The utility of Euler's interior ballistics consequently helped convince virtually the entire European military community of the late eighteenth century (even the Ottoman Turks), to incorporate the analytical calculus and rational mechanics in the education of their artillery officers by the 1790s. One may thus conclude

that Euler was an integral member of the strategic mathematical fraternity that commenced with Archimedes in the Hellenistic era, was revived by Tartaglia, Guidobaldo del Monte and Galileo during the Italian Renaissance, and achieved such terrifying heights with von Neuman and Wohlstetter during the Cold War. [42, p. 299].

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    Reviewed by Clifford Truesdell, MR0107586. The first chapter deals with the history of exterior ballistics from 1700 to 1760 and is the first detailed account of theoretical in ballistics from that period. We owe a great deal of thanks to Roger Cooke for translating portions of this work for us.

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> "Consider, for instance, the surprise of Giovanni Renieri, a gunner who attempted to apply Galileo's theory to his craft, who when he complained in 1647 to Torricelli that his guns did not behave according to Galileo's prediction, was told by Torricelli that "his teacher spoke the language of geometry and was not bound by any empirical result." (Michael Segre, *In the Wake of Galileo*, 1991, p. 43)." Quoted from p. 245 of the book, part of which is on line.

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> Steel writes that this "appears to be the earliest artillery book that assumed a knowledge of the calculus" [40, p. 187], but, other than mentioning fluxions once in the preface ["how can the velocities of shots and shells be determined without being acquainted with the laws of resistance, and which cannot be known without the use of fluxions, nor the finding the curve described by the shot, which is one of the most intricate cases?" (pp. vi–vii)], no evidence to support this claim could be found. This quotation might indicate that he had looked at Euler, for there is a similar sentiment is expressed there. Muller did write about fluxions in some of his other books, books that were used at Woolwich.

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    "The 1805 edition reproduces the 1742 edition word for word, contains the valuable preface to the 1761 edition by James Wilson, and has significant additions and corrections by Charles Hutton, and reproduces several of Robins's other gunnery tracts in their enirety." [From the review of [36] by Howard].

36. Robins, Benjamin, *New principles of gunnery (1724)*, with a new introduction by W. S. Curtis. Richmond, Surrey: Richmond Publishing Company, 1972, pp. lvii + 119.

    Reviewed by Robert A. Howard, *Technology and Culture*, Vol. 15, no. 1. (Jan., 1974), 95–97. This volume consists of a four-page introduction by Curtis, a two-page summary of the contents, a photo-reproduction of the 1742 edition of the *New Principles of Gunnery* [30] and two photo-reproductions of portions of Hutton's 1805 edition of Robins [35], namely, pp. 206–215 and 328-341.

37. Rose, Paul-Lawrence, "Galileo's theory of ballistics," *The British Journal for the History of Science*, Vol. 4, No. 2 (Dec., 1968), pp. 156-159.

38. Spaulding, Thomas M., "Early military books in the Folger Library," *The Journal of the American Military History Foundation*, Vol. 1, No. 3 (Autumn, 1937), pp. 91–100.

39. Steele, Brett D., “Muskets and Pendulums: Benjamin Robins, Leonhard Euler, and the Ballistics Revolution,” *Technology and Culture*, Vol. 35, No. 2 (Apr., 1994), pp. 348–382.

40. Steel, Brett D., *The Ballistics Revolution: Military and Scientific Change from Robins to Napoleon*, Ph.D. dissertation, University of Minnesota, directed by Edwin J. Layton, Jr., 1994.

41. Steel, Brett D., “Military “Progress” and Newtonian Science in the age of Enlightenment,” pp. 361–390 in *The heirs of Archimedes: science and the art of war through the Age of Enlightenment*, edited by Brett D. Steele and Tamera Dorland, Cambridge, Mass.: MIT Press, c2005.

42. Steel, Brett D., “Rational mechanics as Enlightenment engineering: Leonhard Euler and interior ballistics,” pp. 281–302 in *Gunpowder, Explosives, and the State: A Technological History*, edited by Brenda Buchanan, Aldershot, England; Burlington, VT: Ashgate, c2006.

43. Tartaglia, Niccolò, *Three bookes of colloquies concerning the arte of shooting in great and small peeces of artillerie, variable randges, measure, and waight of leaden, yron, and marble stone pellets, minerall saltepeeter, gunpowder of diuers sortes, and the cause why some sortes of gunpower are corned, and some sortes of gunpowder are not corned: written in Italian, and dedicated by Nicholas Tartaglia vnto the Royall Prince of most famous memorie Henrie the eight, late King of England, Fraunce, and Ireland, defender of the faith &c. And now translated into English by Cyprian Lucar Gent. who hath also augmented the volume of the saide colloquies with the contents of euery colloquie, and with all the corollaries and tables, that are in the same volume. Also the said Cyprian Lucar hath annexed vnto the same three books of colloquies a treatise named Lucar Appendix*, London, 1588.

The full text is available via Early English Books Online.

44. Tempelhof, Georg Friedrich von (1737–1807), *Anfangsgründe der Analysis des Unendlichen: erster Theil, welcher die Differential-Rechnung enthält: zum Gebrauch der Königlichen Preussischen Artillerie entworfen*, Berlin and Stralsund, 1770.

> Despite the fact that this book was written for the use of the Royal Prussian Artillery, there is nothting in this work related to artillery or ballistics. Perhaps this was to be included in the second part, but this was all that was published. Although I saw no references to Euler, his influence could be seen at several places.
> We need to look at more calculus books written for artillery officers or military schools before 1800 and see what they contain about projectile motion. Perhaps this material is in mechanics books, as people who took calculus usually also took mechanics.