QUALITATIVE GRAPHING TECHNIQUES

In calculus you will find that you need to graph many functions. We present here a method of graphing functions which is quite easy to apply and which does apply to many of the functions you will encounter in a calculus course. This method is called "qualitative" because it gives the salient qualities of a graph: it gives a general idea as to what the curve looks like. A more precise graph—the fine-tuning of the graph—can then be obtained by using the methods of the calculus. The techniques presented here do not require the calculus for their use, but they do require it for justification. Finally, we can promise that a mastery of these techniques will make your study of the calculus a good deal easier. It also will give you a good feeling for graphs which is useful in many fields.

V. Frederick Rickey

Polynomials

A polynomial is a function of the form

$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ with $a_n \neq 0$. For example, $3 + 2x - x^2$ and $x^6 + 4x^2$ are polynomials, but $3 - x^{-1}$ and $\sqrt{x}$ are not. The Fundamental Theorem of Algebra says that every polynomial can be factored as a product of linear and irreducible quadratic factors. Linear factors are those of the form $x + c$, such as $x + 5$ and $x - 3$. $x^2 + 1$ is an example of an irreducible quadratic; it is of the second degree and cannot be factored (try the quadratic formula).

Example 1. $x^5 - 14x^4 + 61x^3 - 65x^2 - 50x - 125 = (x^2 + x + 1)(x - 5)^3$. This polynomial of the fifth degree has been factored into an irreducible quadratic and the cube of a linear term. To see that $x^2 + x + 1$ is irreducible apply the quadratic formula to the equation $x^2 + x + 1 = 0$:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2}$$

and observe that we obtain $\sqrt{-3}$, which is not a real number.

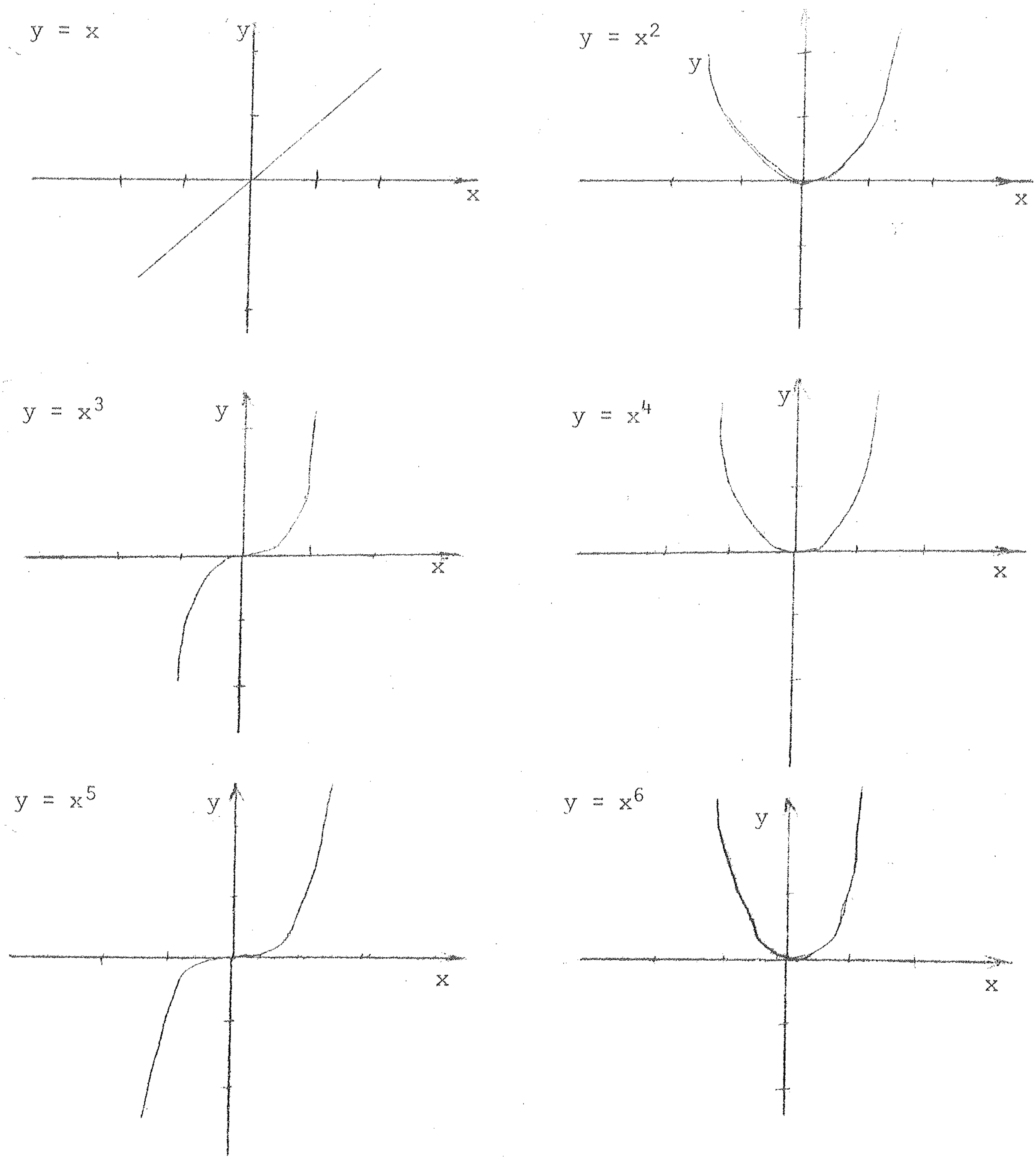
We are interested only in polynomials which are completely factored into linear factors. In fact, the methods we develop below will apply only to polynomials which have been completely factored. We are not trying to factor polynomials (in general, this can be very hard).

Example 2. $x^3 - 13x^2 - 8x + 12 = (x + 3)(x - 2)^2$ is completely factored into linear factors. Observe that this is not the case in Example 1.

We are not claiming that all polynomials can be completely factored into linear factors, but all polynomials which we shall consider here can be.

The basic building blocks of all polynomials are the powers x^n, so we begin by graphing them:

BASIC GRAPHS I.



There are several important things which we should observe about these graphs.

1) They all go through the points (0,0) and (1,1).

2) The graphs of the odd powers of x go through the point (-1,-1), while those of even power go through (-1,1).

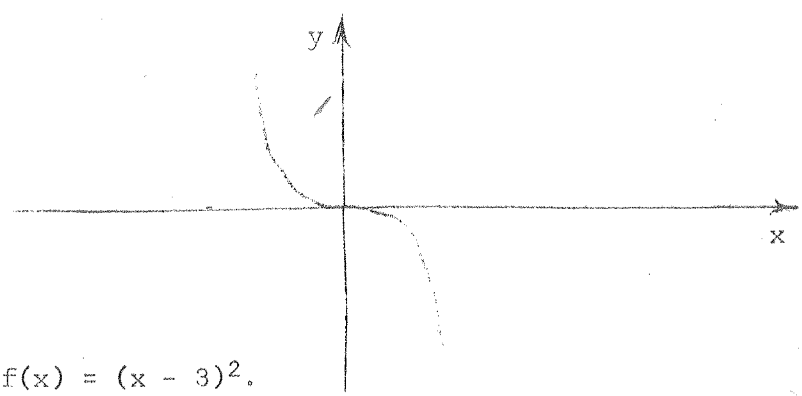
3) If the power is odd then the graph crosses the x-axis, while if it is even the graph touches the x-axis.

4) The higher the power of x is, the "flatter" the graph is at the origin.

Now we shall introduce two variations of these basic graphs.

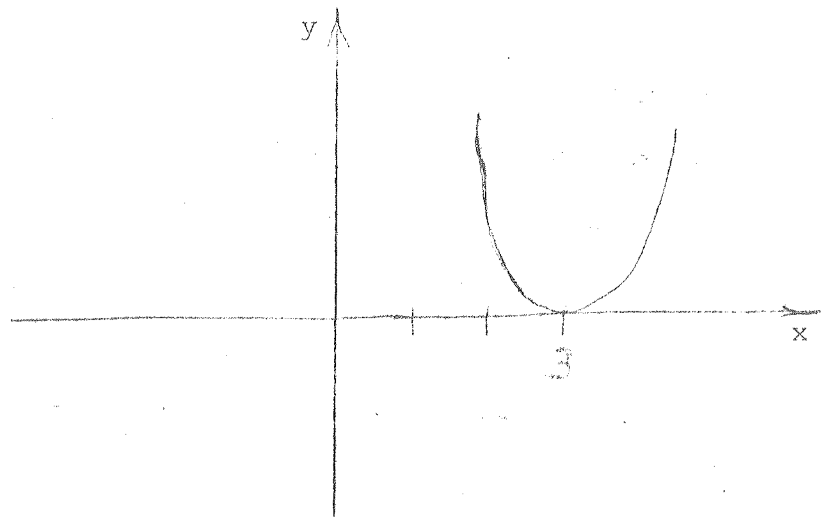
Example 3. $f(x) = -x^3$.

This graph looks just like x^3 except for the minus sign. The minus sign flips the graph over so that we have:



Example 4. $f(x) = (x - 3)^2$.

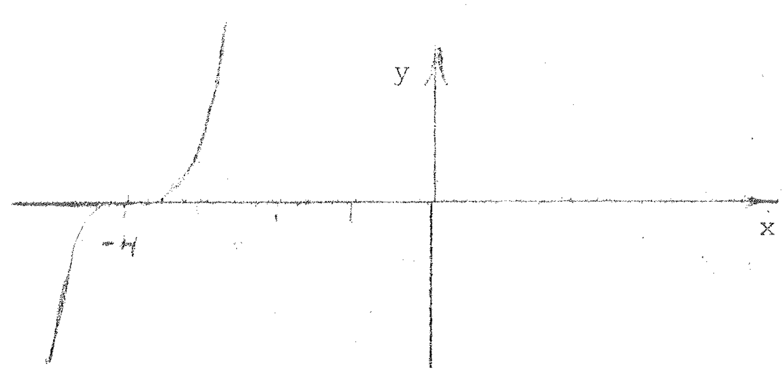
This resembles the graph of $y = x^2$ except that its graph touches the x-axis at (0,0) while the graph of $y = (x - 3)^2$ touches it only at (3,0). Thus the graph is



In general, the graph of $y = (x - a)^n$ looks just like the graph of $y = x^n$ except that it crosses or touches the x-axis at the point $(a,0)$ instead of at $(0,0)$.

Example 5. $f(x) = (x + 4)^3$.

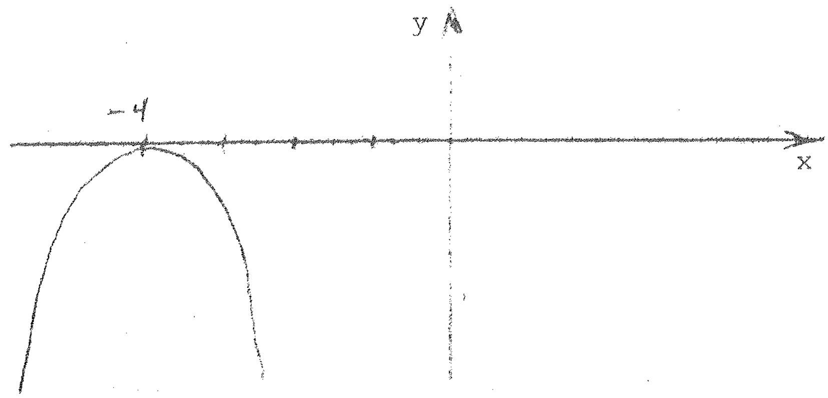
The graph looks like that of $y = x^3$ except that it crosses the x-axis at $(-4,0)$.



The main thing to note here is that the plus sign moved the graph to the left while the minus of Example 4 moves it to the right.

Example 6. $f(x) = -(x + 4)^2$.

There are two things to note here. The minus sign makes the graph look like that of $y = -x^2$, while the $+4$ moves the graph 4 units to the left.



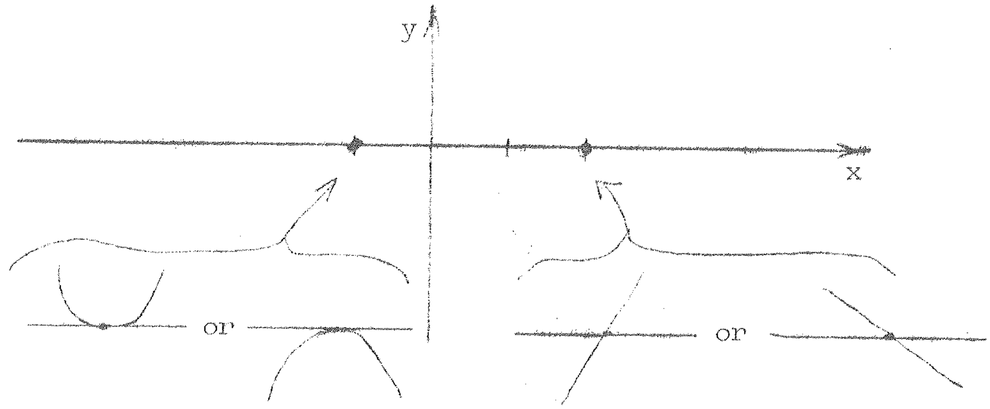
Now we come to the MAIN METHOD: If we have a polynomial which has been completely factored into linear factors, say,

$$f(x) = c(x - a_1)^{n_1}(x - a_2)^{n_2}\ldots(x - a_k)^{n_k} ;$$

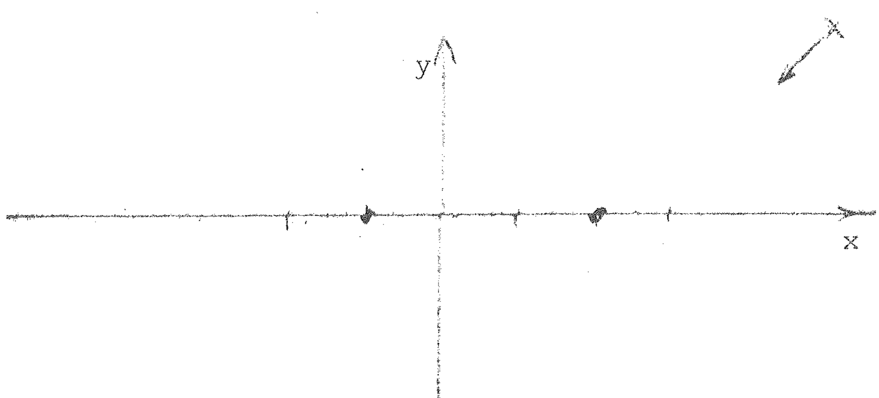
(where $c, a_1, a_2, \ldots, a_n$ are real numbers and $n_1, n_2, \ldots, n_k$ are positive integers), then the graph of f crosses or touches the x-axis at the points $(a_i, 0)$ (because $f(a_i) = 0$) and at no other points. Near the point $(a_i, 0)$ the graph looks like the graph of $y = \pm x^{n_i}$. Moreover, the whole graph is nice and smooth, without breaks and jumps (continuous is the technical word). Also there are no unnecessary humps or wiggles in the curve.

Example 7. $f(x) = (x - 2)(x + 1)^2$.

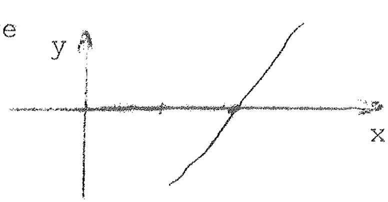
The first thing to observe is that this curve intersects the x-axis in two points, when x is 2 and when x is -1 and at no other points. The points are called the zeroes of the function. At the point $(2,0)$ the graph looks like the graph of $y = \pm x$ and at $(-1,0)$ it looks like the graph of $y = \pm x^2$. Thus we have:



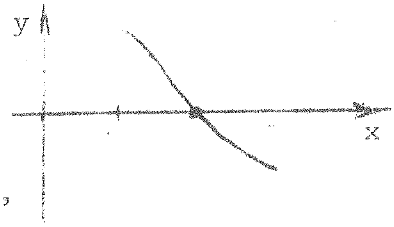
Now to decide which of the four possibilities is correct we must select one more point, but it doesn't matter which one. Really all we care about is the sign of $f(x)$ for some x (different from -1 and 2). Say you put in 1,000,000 for x. Then $x - 2$ and $x + 1$ are both positive and the product is positive, so $f(1{,}000{,}000)$ is positive. We put a cross on the graph to indicate our "starting point" and then proceed to draw the curve from right to left:

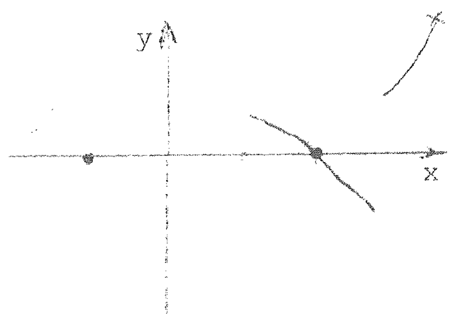


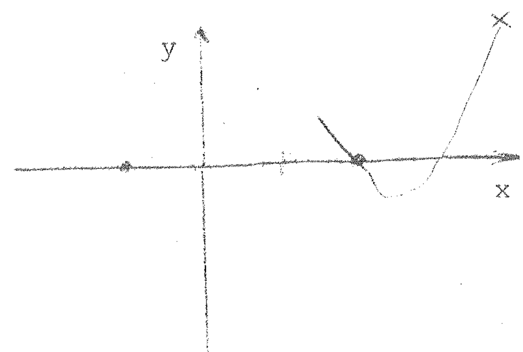
As we come down from our starting point to go through the point $(2,0)$ we have to make the graph look like



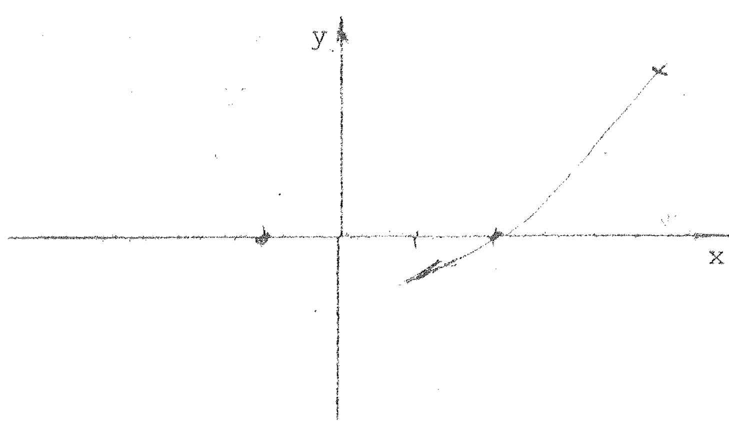
There is no way to make it look like without having a break in the graph or having the graph go below the x-axis, i.e., neither of the following is possible:



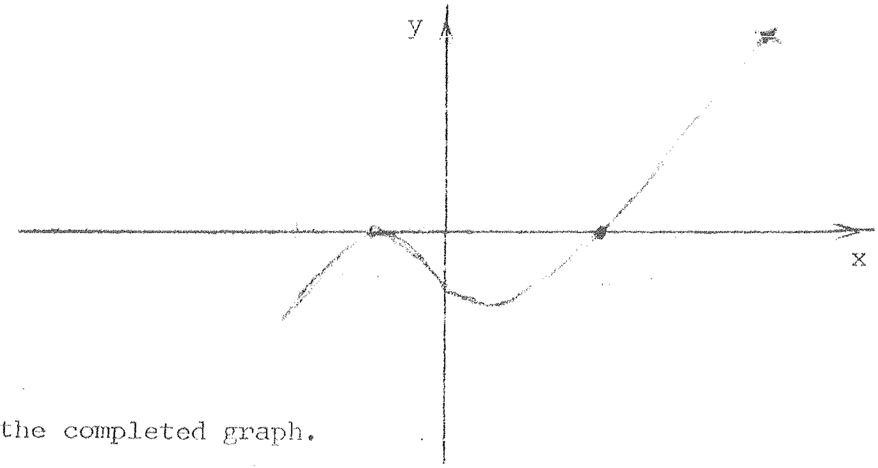




The first of these puts a break in the graph while the second puts in a third zero. So the graph must continue as follows:



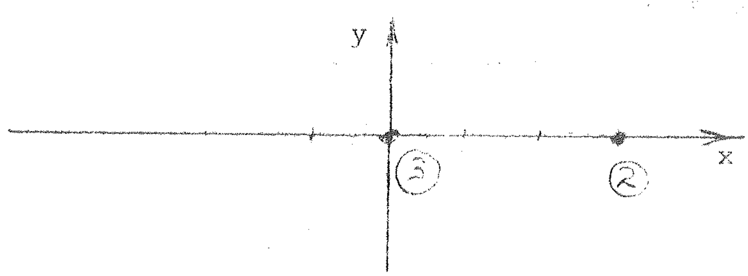
Now it must come back up and touch the axis at (-1,0) and then head down again.



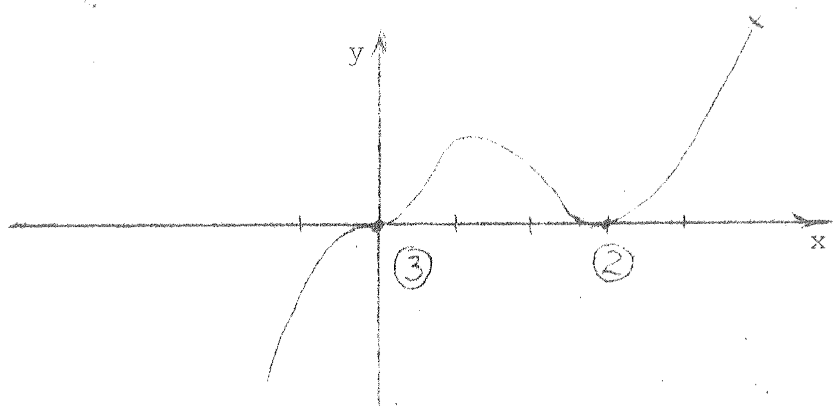
This is the completed graph.

Example 8. $f(x) = x^3(x - 3)^2$.

First we plot the zeroes and below each in a little circle we write the multiplicity, i.e., the exponent associated with that zero. In this case the multiplicity of the zero 0 is 3 and the multiplicity of the zero 3 is 2. Thus we have:



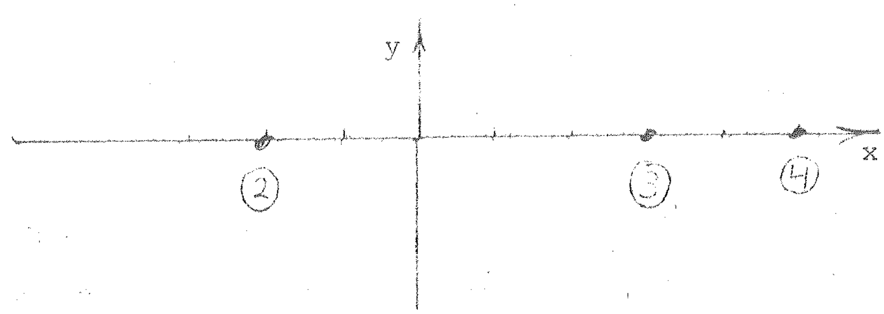
The multiplicity 3 tells us that at (0,0) the graph looks like that of $y = \pm x^3$ and the multiplicity 2 tells us that at (3,0) the graph looks like that of $y = \pm x^2$. Now again we find a starting point. If x is very large then x^3 and $(x - 3)^2$ are large and positive and so the value of the function is large and positive. We put a cross in the picture to indicate this starting point and then draw the graph from right to left.



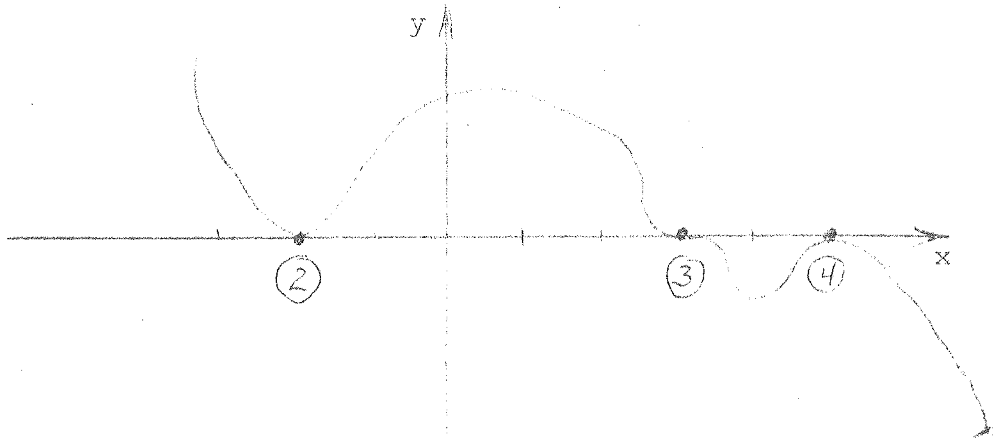
Observe that at (3,0) the graph looks like that of $y = x^2$ and that (0,0) it looks like that of $y = x^3$. Between the two zeroes we were forced to put in a hump but there are no extra ones.

Example 9. $f(x) = -(x + 2)^2(x - 3)^3(x - 5)^4$.

First plot the zeroes and their multiplicities:



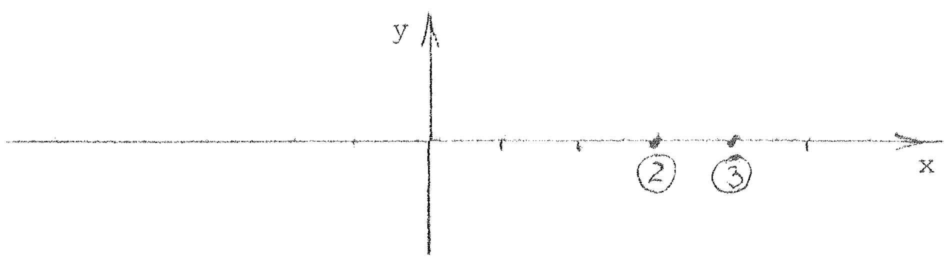
Then plot a starting point: If x is very large, say 1,000,000, then each of $x + 2$, $x - 3$, and $x - 5$ is positive. Thus $(x + 2)^2(x - 3)^3(x - 5)^4 > 0$ and $f(x) < 0$. So we start at the cross and draw the graph:



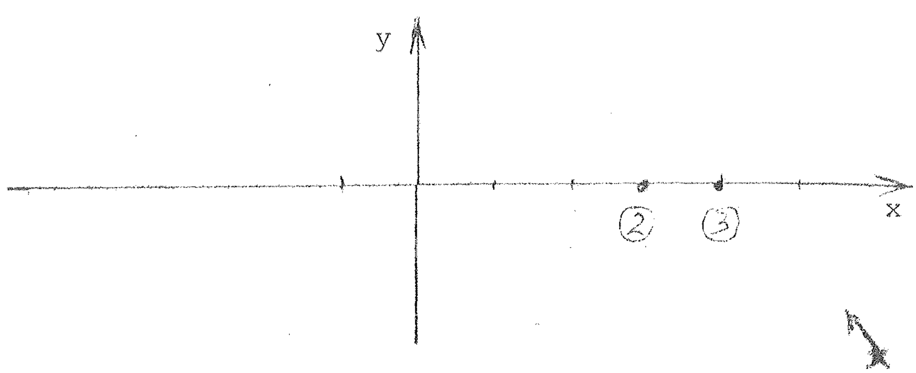
If one wants to check this graph one can plot a few additional points.

Example 10. $f(x) = (x - 3)^2(4 - x)^3$.

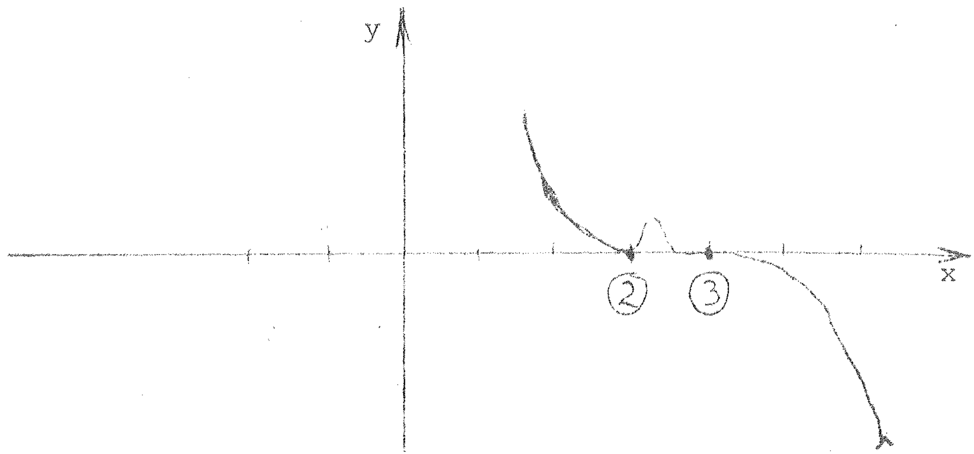
We see that there is a zero of multiplicity 2 at 3 and one of multiplicity 3 at 4. So we have:



Now to find a starting point let x be 1,000,000. Then $(x - 3)^2$ is large and positive. But $4 - x$ is large and negative. Its cube is also negative. Thus f(1,000,000) is large and negative as is indicated by the cross:



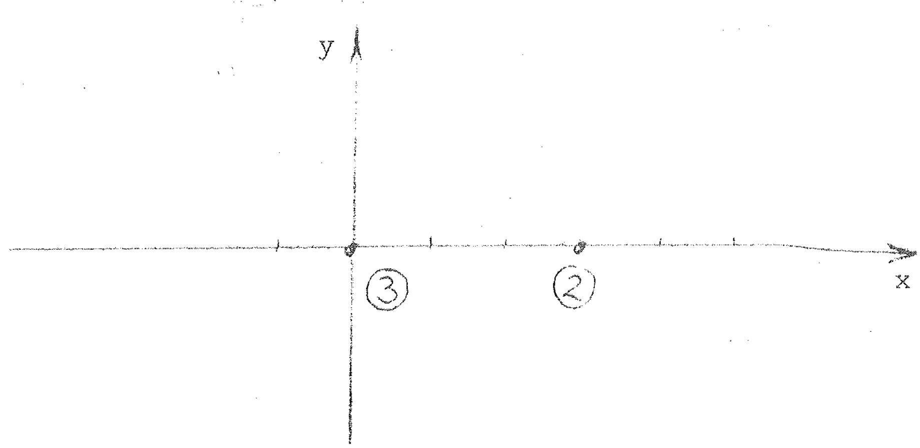
When we draw in the graph taking into account the starting point, zeroes and their multiplicities we get:



Now we shall discuss a different way to sketch the graph of a completely factored polynomial. The following way can be used to solve certain types of inequalities, so both techniques should be mastered.

Example 11. $f(x) = x^3(x - 3)^2$.

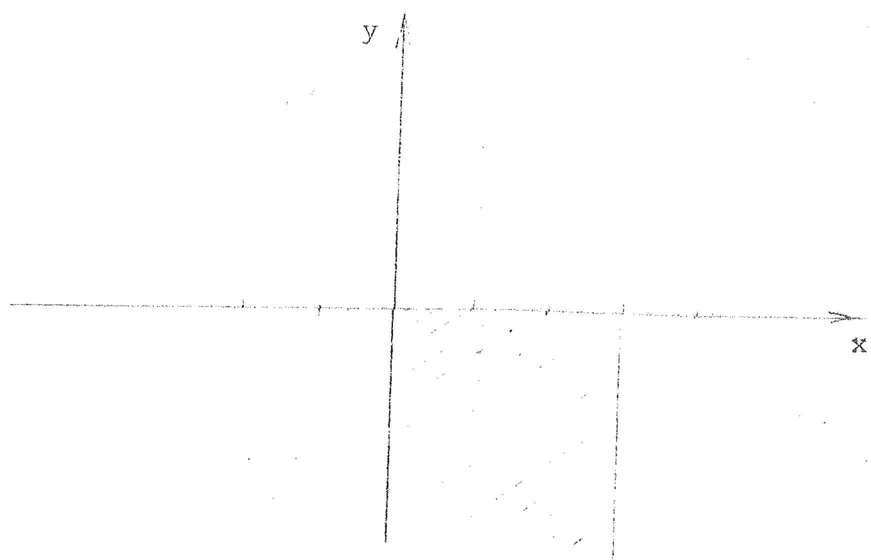
First we plot the zeroes and below each in a little circle we write their multiplicities, i.e., the exponents:



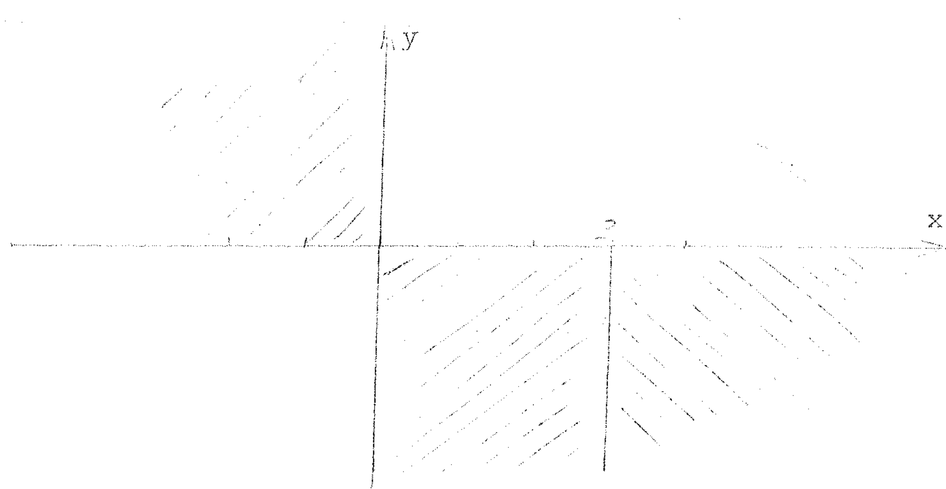
Now we could plot a "starting point" as in the above examples, but we shall instead look at the extent of the graph. We shall shade in those regions where the graph isn't, i.e., the so-called excluded regions. There are two zeroes and they divide the line into three intervals:

$$(-\infty, 0), \qquad (0,3), \qquad (3,\infty) .$$

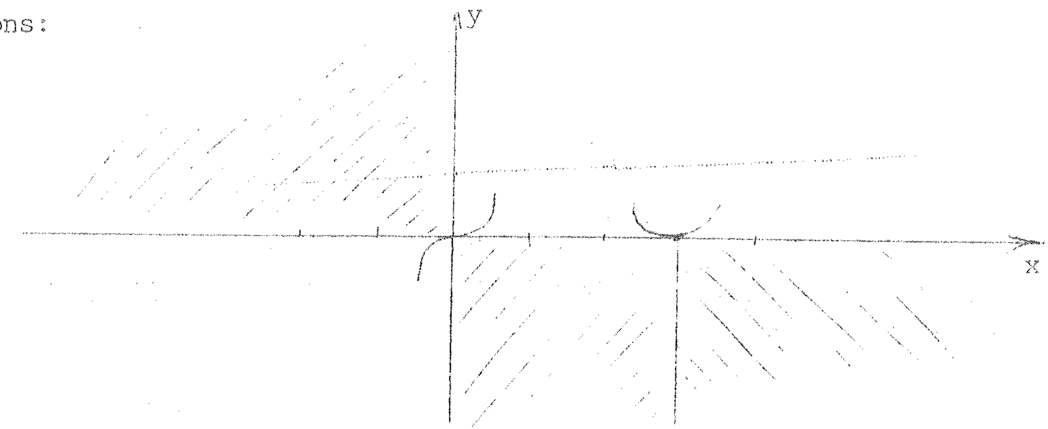
In each of these intervals the graph is either entirely positive or entirely negative (continuity), so we need only look at one point in each interval to decide which. If $0 < x < 3$, then $x^3 > 0$ and $(x - 3) < 0$ but $(x - 3)^2 > 0$, so $f(x) > 0$. Then we shade out the region below the interval $(0,3)$:



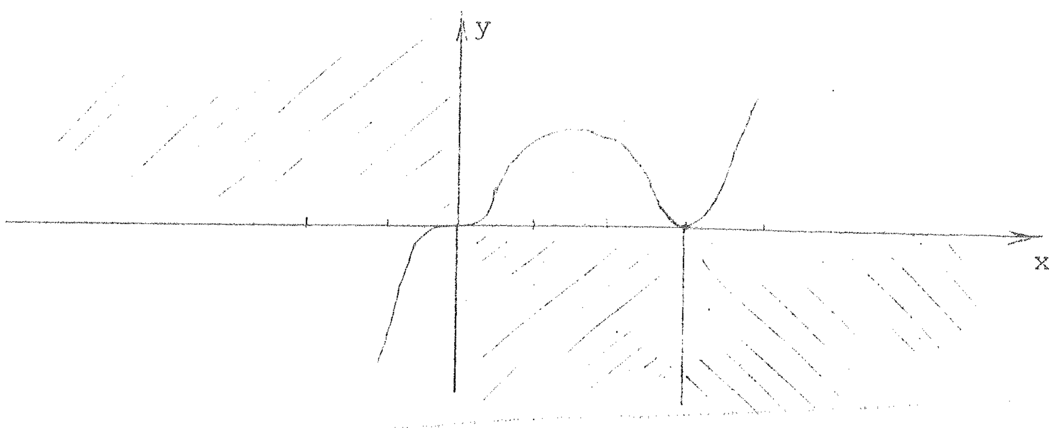
Similarly $f(x) > 0$ if $x > 3$ and $f(x) < 0$ if $x < 0$ so we have



Now at (0,0) the graph must look like that of $y = x^3$ and at (3,0) it must look like that of $y = x^2$ as the graph cannot go into the shaded regions:

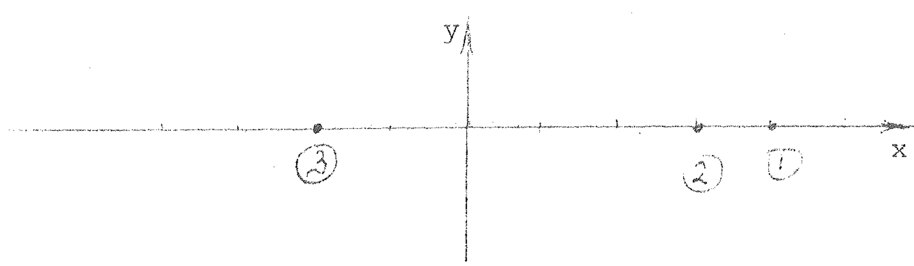


Now when we connect these pieces with a smooth graph, without any extra bumps or wiggles, we get the finished graph.



Example 12. $f(x) = (x + 2)^3(x - 3)^2(x - 4)$.

First plot the zeroes and their multiplicities.



To check the extent we note that the three zeroes break the line into four intervals: $(-\infty,-2)$, $(-2,3)$, $(3,4)$, $(4,\infty)$. Now we must check the sign of the function in each of these intervals. The bookkeeping is easily handled via the following chart:

on $(-\infty,-2)$: $- + - = +$

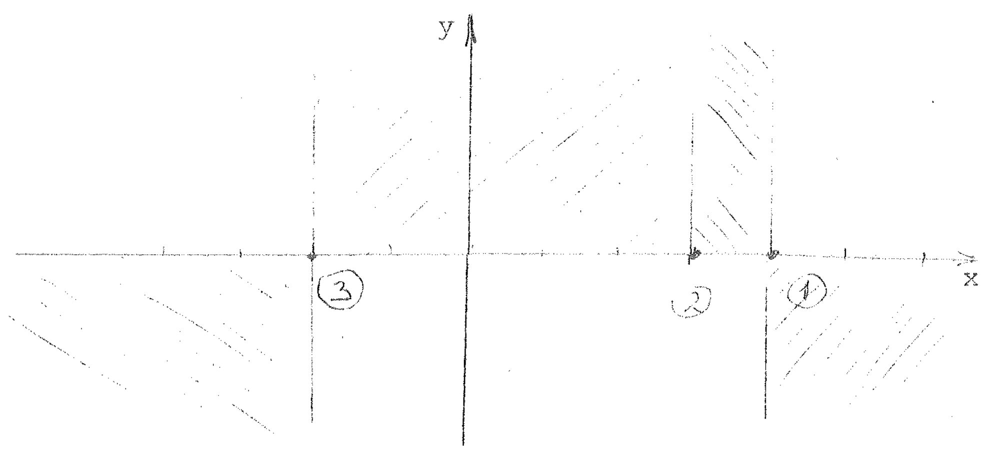
on $(-2,3)$: $+ + - = -$

on $(3,4)$: $+ + - = -$

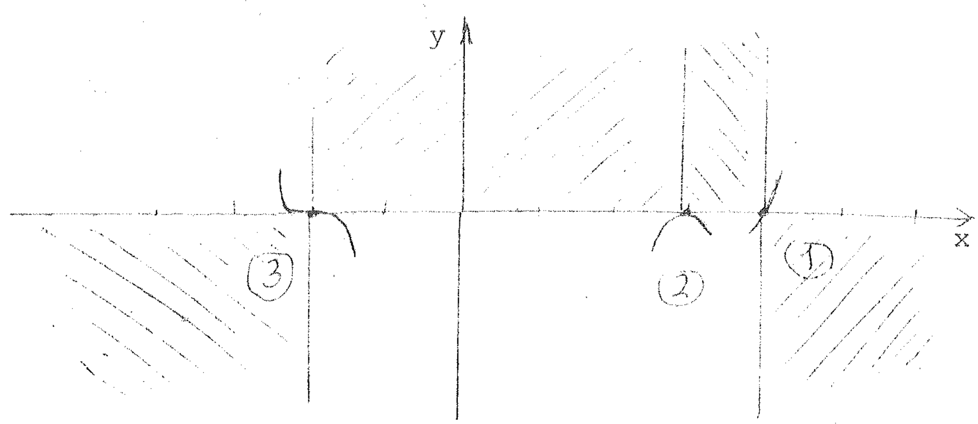
on $(4,\infty)$: $+ + + = +$

On the interval $(-\infty,-2)$ the $- + -$ means that when $x < -2$ then $(x + 2)^3$ is negative, $(x - 3)^2$ is positive and $(x - 4)$ is negative. The product of these is positive.

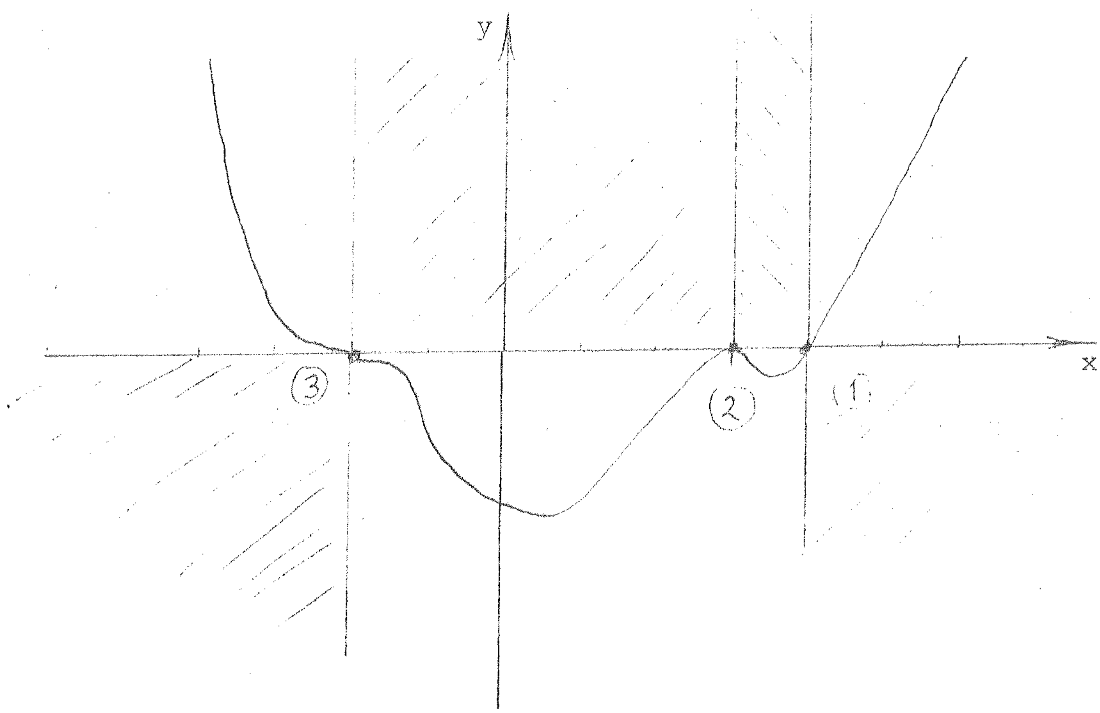
Using this information we can now shade the graph:



Then we can draw in the curve near where it crosses the axis using the multiplicity and by staying out of the excluded regions:



Finally we can complete the graph:



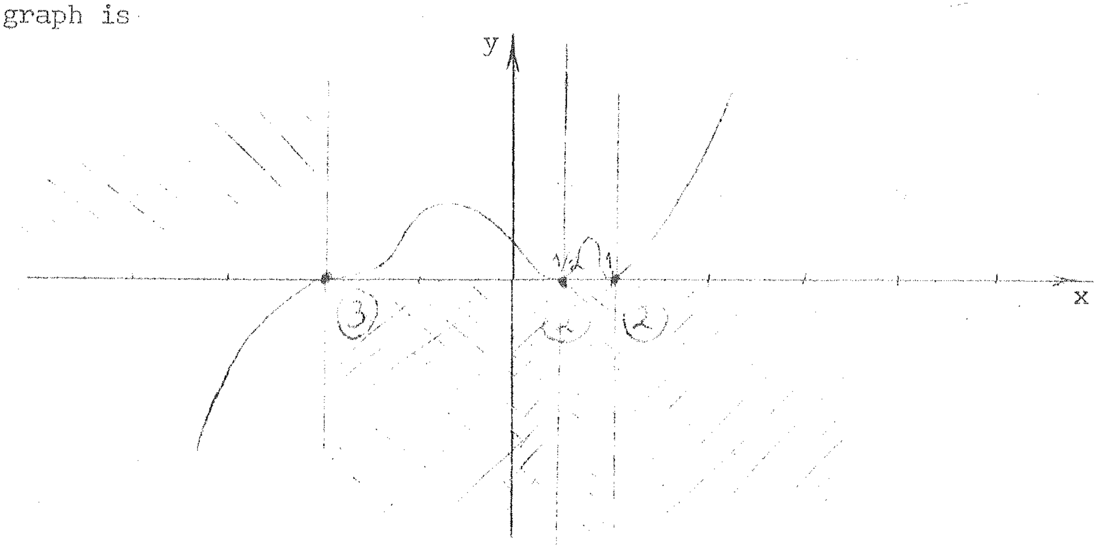
Note that we were forced to draw the minimum points or valleys in the intervals (-2,3) and (3,4). We do not know the location of these minimum points, in fact, we shall use the calculus in order to locate them exactly.

You should also observe that in all of the examples we never put in more wiggles or bumps than we are forced to. Using the calculus one can prove that if one puts in no more wiggles or bumps than necessary then the graph is qualitatively correct.

Example 13. $f(x) = (x - 1)^2(x + 2)^3(2x - 1)^2$.
Here the factor $(2x - 1)$ looks unusual. It is not of the form $x - c$, but can be made that way by factoring out the 2 to get $2(x - 1/2)$. Thus

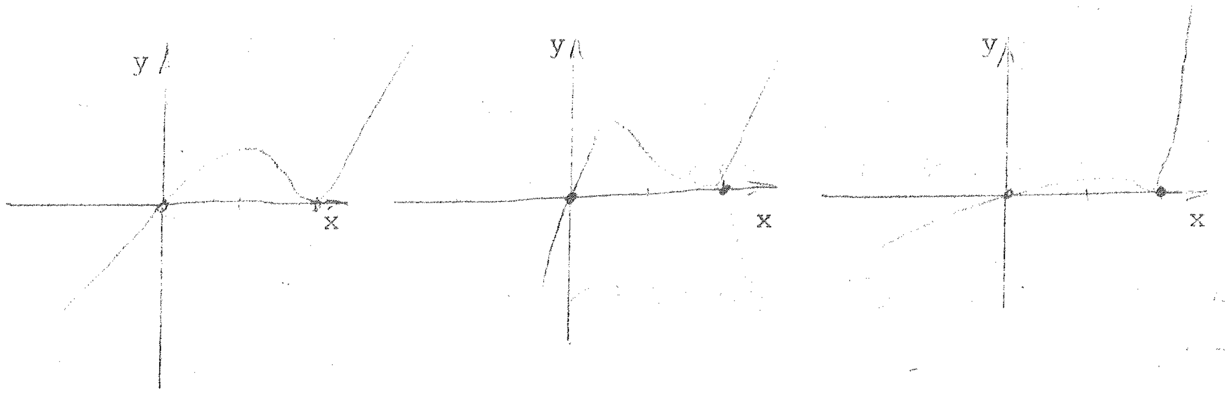
$$f(x) = 4(x - 1)^2(x + 2)^3(x - 1/2)^2.$$

The graph is



The factor 4 in the function f does change the graph (it stretches it, in the vertical direction), but it does not change the qualitative nature of the graph, i.e., the graph of $g(x)$ and $4g(x)$ have the same zeroes, multiplicities and general shape. Because we do not care here about how high up or down the curves go we have not placed any scale on the y-axis.

At this time we can more carefully explain what we mean by the word "qualitative." Take the simple example $f(x) = x(x - 2)^2$. Now qualitatively all of the following graphs are correct:



The exact location of the maximum point (the top of the bump between the zeroes) is a quantitative feature of the graph. To locate it precisely one needs to use the calculus. Our methods just give the general shape of the graph. Further details can be obtained using the calculus.

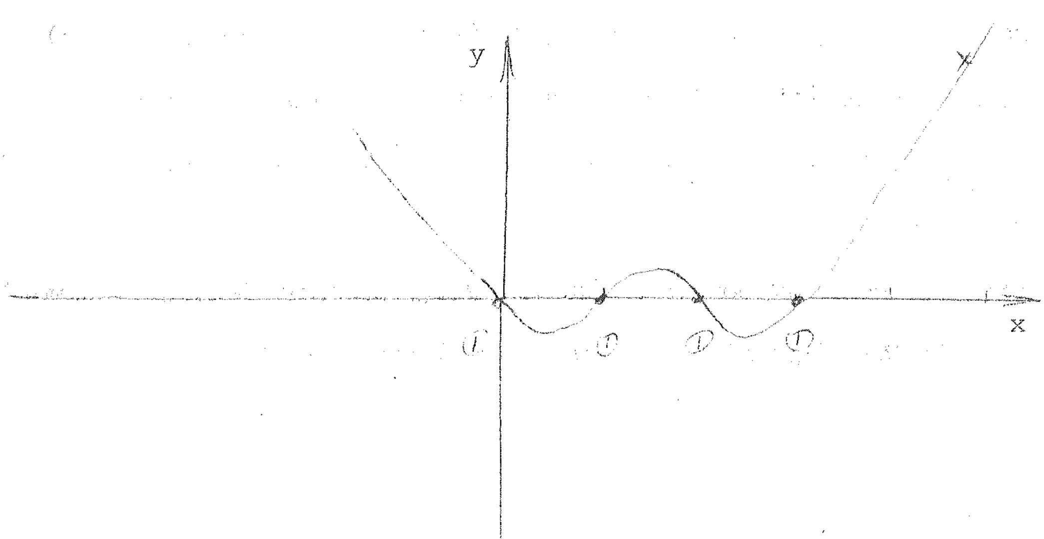
We shall conclude with several additional examples without comments. The general plan is always:

1) plot the zeroes and their multiplicities;

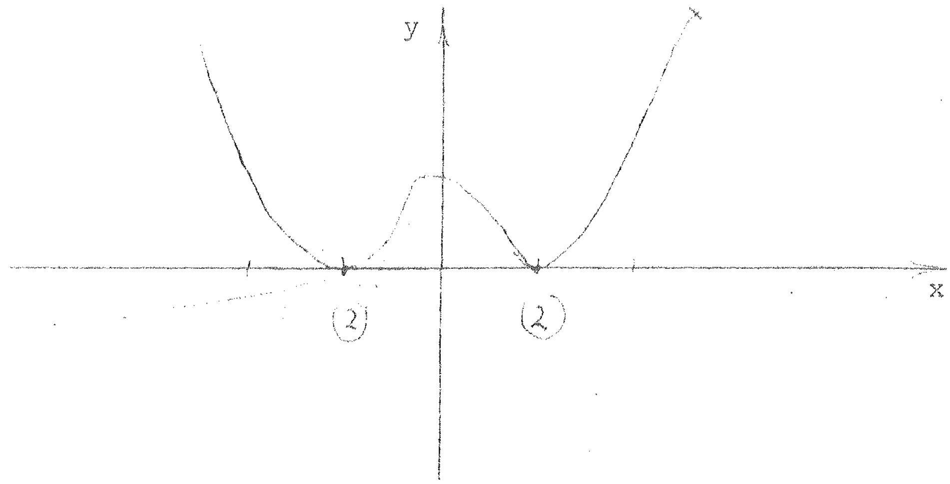
2) find a starting point (or shade out the excluded regions);

3) draw a smooth graph utilizing the above information (and don't put in any extra bumps or wiggles).

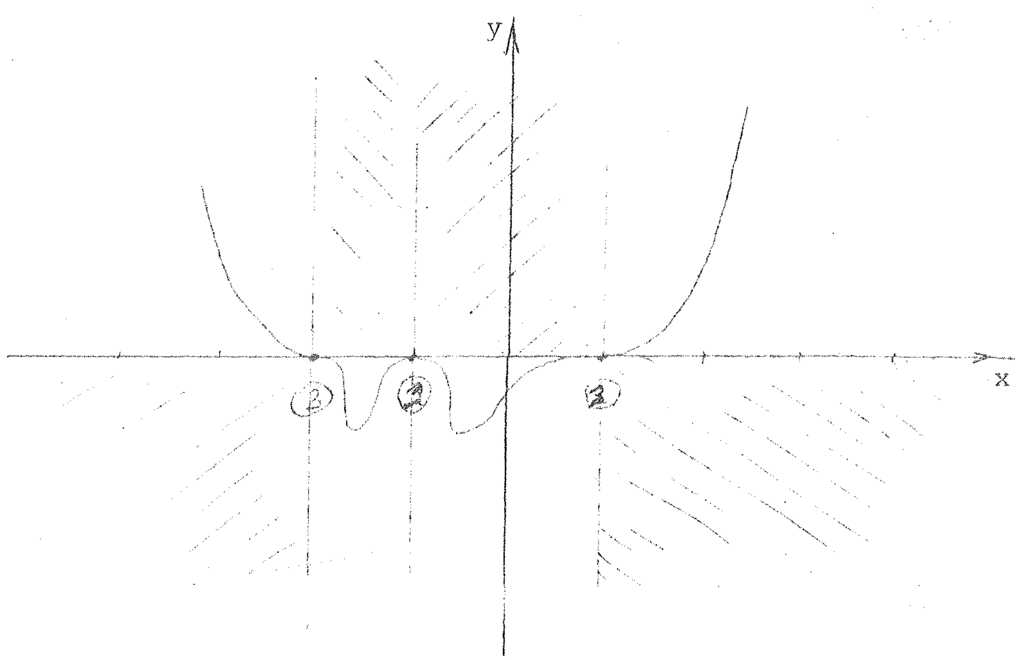
Example 14. $f(x) = x(x - 1)(x - 2)(x - 3)$.



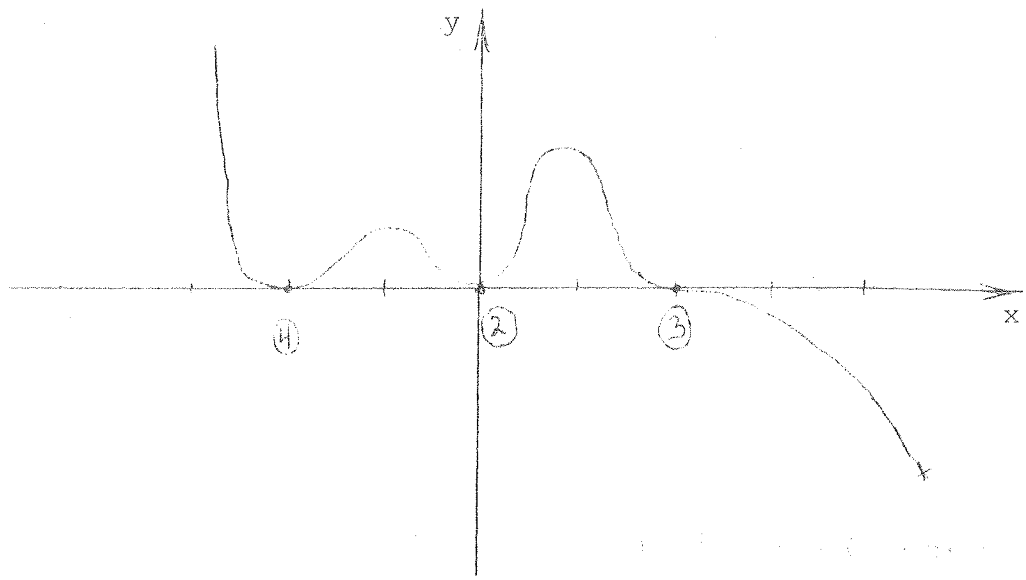
Example 15. $f(x) = (x - 1)^2(x + 1)^2$.



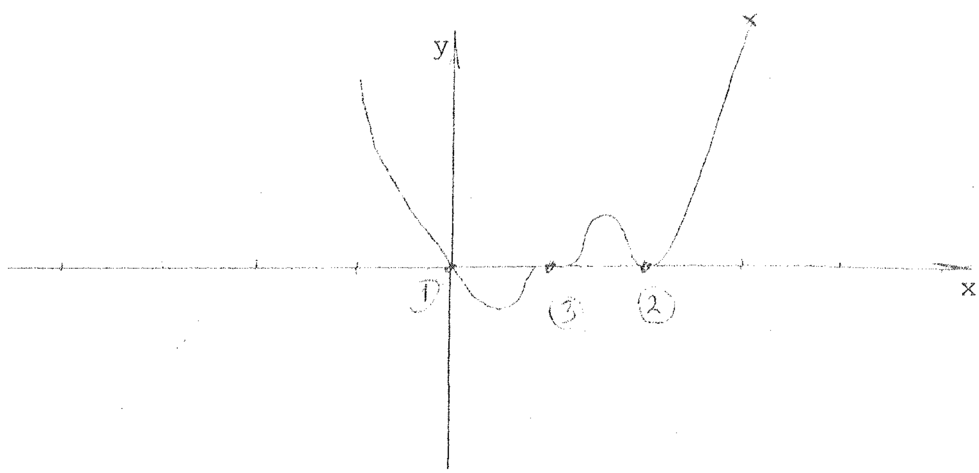
Example 16. $f(x) = (x + 1)^2(x - 1)^3(x + 2)^3$.



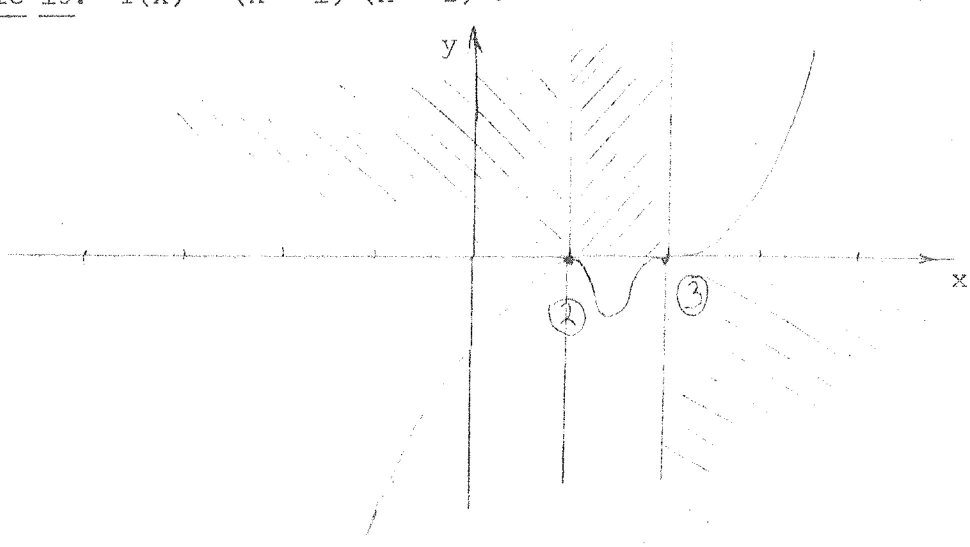
Example 17. $f(x) = -(x - 2)^3(x + 2)^4x^2$.



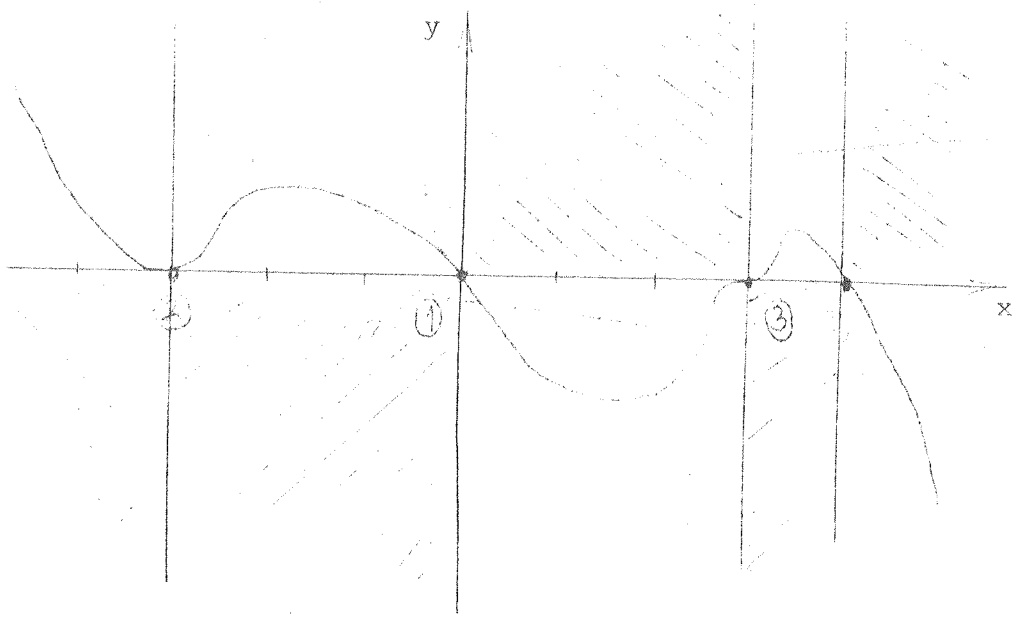
Example 18. $f(x) = (x - 1)^3(x - 2)^2(x)$.



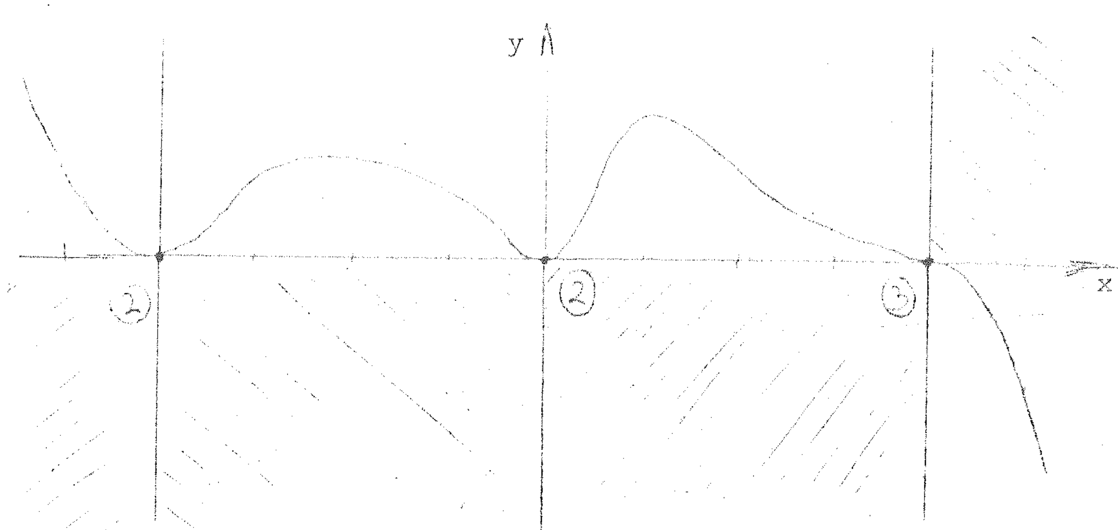
Example 19. $f(x) = (x - 1)^2(x - 2)^3$.



Example 20. $f(x) = x(3 - x)^3(x - 4)(x + 3)^2$.



Example 21. $f(x) = -3x^2(x - 4)^3(x + 4)^2$.



Hopefully by this time you see that this is really quite easy, no matter which method you use. So it is time for you to do some on your own.

EXERCISES:

Part I

Graph each of the following polynomials. Each root or zero should be marked with a small circle indicating its multiplicity; for example, a zero of multiplicity three should be indicated thus: (3). You should pay particular attention to the shape of the graph around the roots. Besides plotting the zeroes, you should plot at most one more point.

1. $f(x) = -x^2$
2. $f(x) = 3x^3$
3. $f(x) = (x - 2)^2$
4. $f(x) = -(x - 2)^3$
5. $f(x) = (3 - x)^3$
6. $f(x) = (x - 1)(x + 1)$
7. $f(x) = (x - 3)(x - 5)$
8. $f(x) = x^2 + 2x - 8$
9. $f(x) = x^2 - 2x + 3$
10. $f(x) = (x - 3)^2(x + 2)$
11. $f(x) = (x - 3)(x + 2)^2$
12. $f(x) = (x - 1)(x + 2)(x - 4)$
13. $f(x) = (2x - 1)(x - 2)(x - 3)$
14. $f(x) = (x - 2)(x + 3)^2(x - 4)^3$
15. $f(x) = (x - 2)^2(x + 3)^2(x - 4)^2$
16. $f(x) = (x + 2)(2x - 1)(x - 4)^2$
17. $f(x) = (x - 6)^2(x + 4)^3(4 - x)^4$
18. $f(x) = (x - 2)^2(x + 3)^3$
19. $f(x) = (x + 4)^3(x - 6)^5$
20. $f(x) = x(x - 1)^2(x + 2)^3$
21. $f(x) = x^3(x - 1)^2$
22. $f(x) = x^4(x + 2)^2$
23. $f(x) = x^2(x - 1)^2(x + 3)^3$
24. $f(x) = x^2(x - 1)^3(x + 3)^2$
25. $f(x) = (x - 3)^3(3x + 1)(2x - 1)^2$
26. $f(x) = 3(x - 3)^3(3x + 1)^2(2x - 1)$
27. $f(x) = x^3 - x^2 - 2x$
28. $f(x) = x^4 + 2x^3 + x^2$
29. $f(x) = x^3 - x^2 - 3x - 6$
30. $f(x) = x^4 - x^2 + 1$

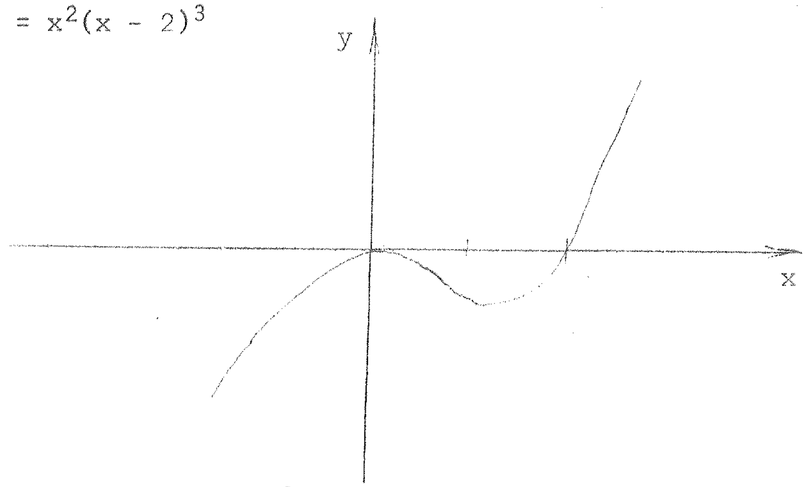
Part II

1. Carefully graph all of the BASIC GRAPHS I on the same set of coordinate axes (colored pencils would help). Plot the points on the functions for the following values of x: 0, ±1/2, ±1, ±3/2. Observe how the graphs are related to one another.

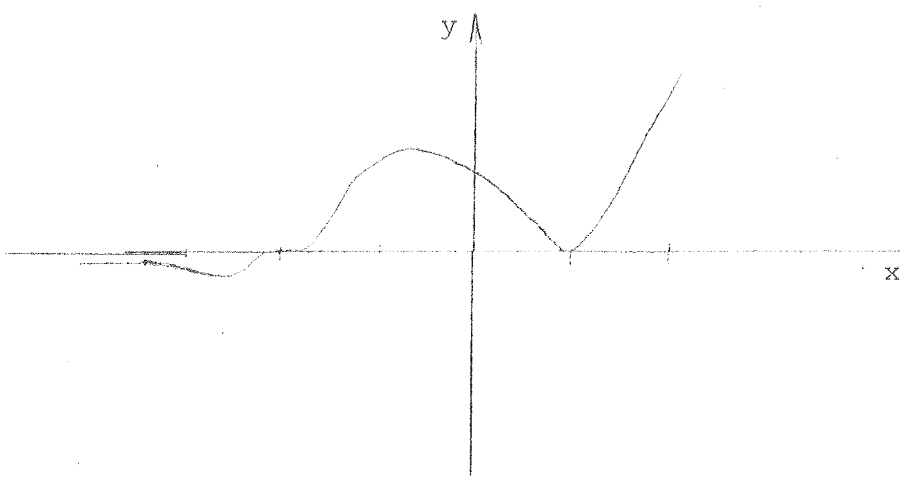
2. On one set of coordinate axes plot $y = x^2$, $y = -x^2$, $y = 2x^2$, $y = (x - 2)^2$. Plot points for $x = 0, \pm 1/2, \pm 1, \pm 3/2$ for the first three and for 1/2, 1, 3/2, 2, 5/2, 3, 7/2 for $y = (x - 2)^2$. Observe the relationships between the grpahs.

3. What is wrong with each of the following graphs?

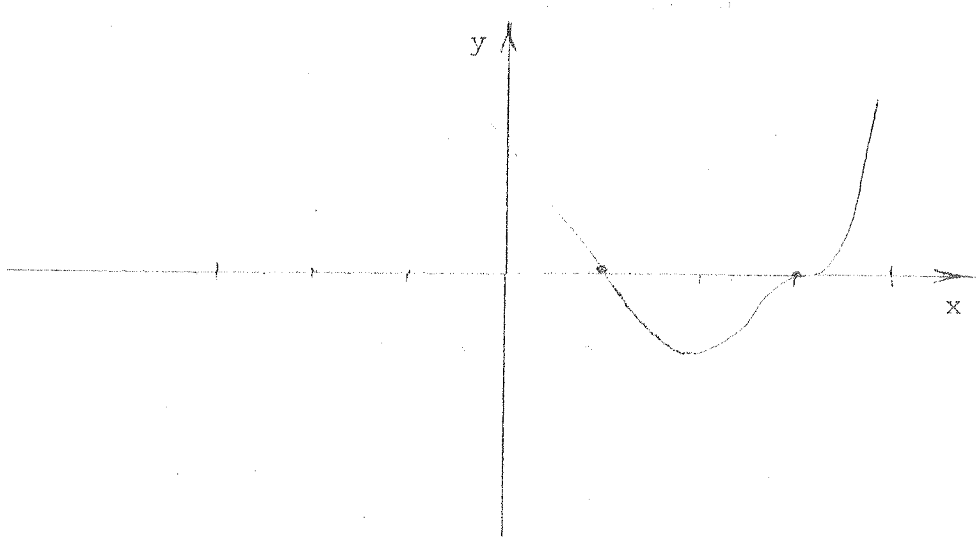
a) $f(x) = x^2(x - 2)^3$



b) $f(x) = (x - 1)^2(x + 2)^3$

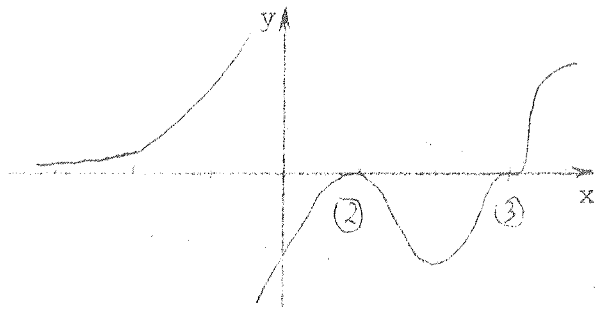


c) $f(x) = (x - 1)(3 - x)^3$

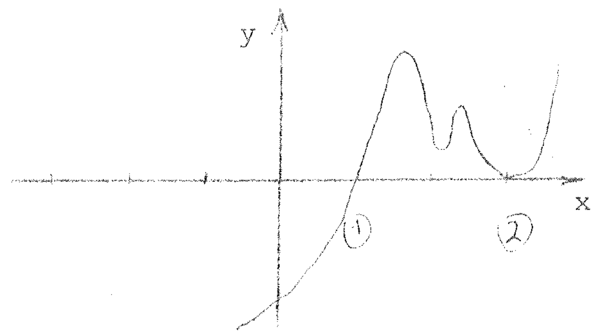


4. Indicate what is wrong with each of the following graphs, which are supposedly graphs of polynomials.

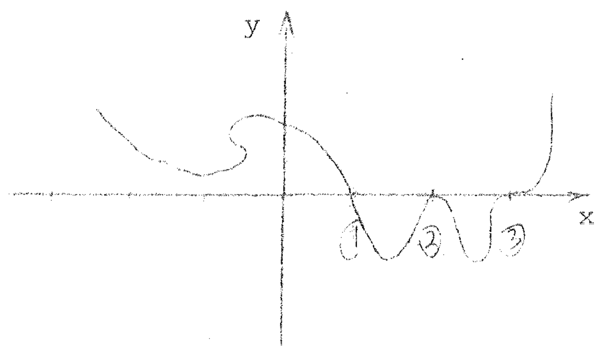
a)



b)

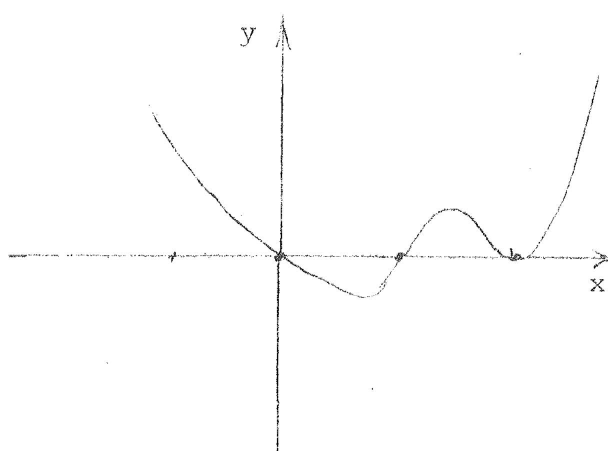


c)

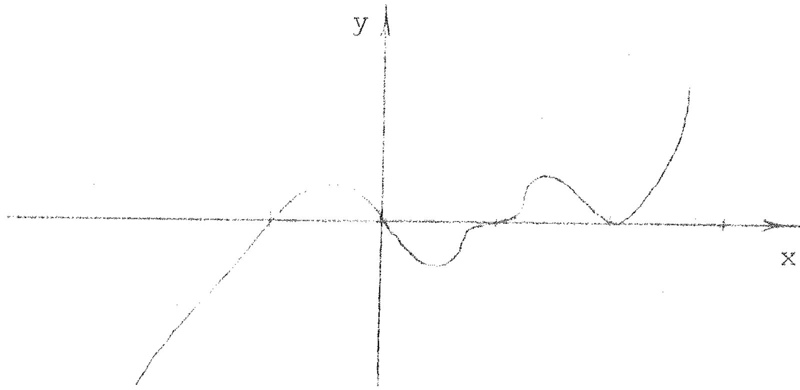


5. Each of the following is the graph of a polynomial. Find the equation of the polynomial which best fits this graph.

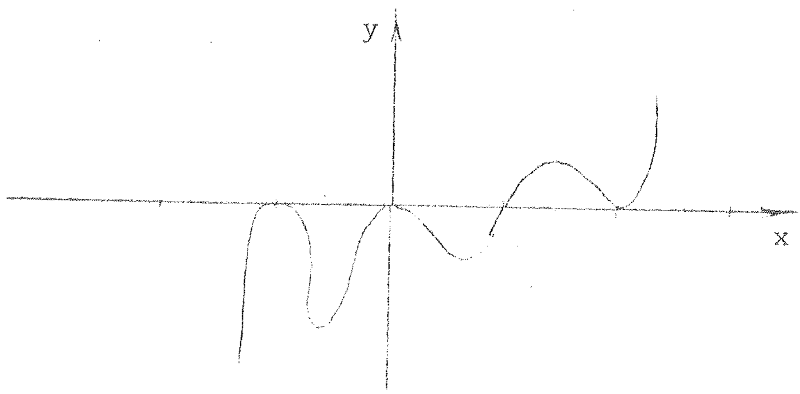
a)



b)

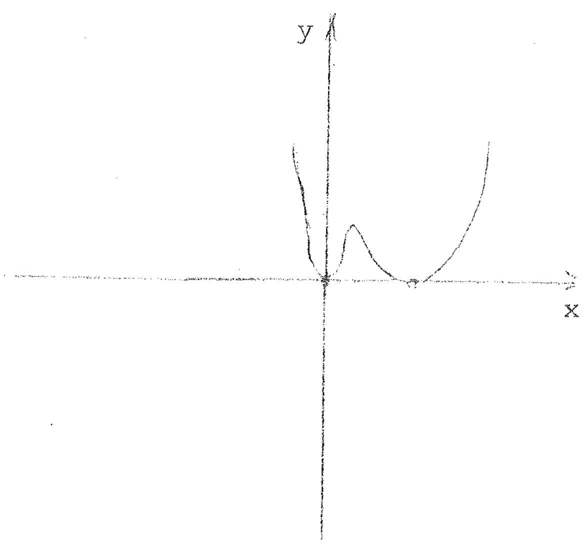


c)



6. Which of the following is the <u>best</u> graph of $y = x^2(x - 1)^4$?

a)



b)