Machin's Formula for Computing Pi

V. Frederick Rickey
Department of Mathematical Sciences
United States Military Academy
West Point, NY 10996
fred-rickey@usma.edu

March 10, 2010

Around 1706, John Machin found an interesting arctangent formula

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

that he expanded using Gregory's arctangent series and then computed π "True to above a 100 Places." We shall prove Machin's formula, discuss how it has been used to compute π, and discuss its interesting history and ramifications. We shall refrain from the "indoor sport" of deriving variants of Machin's formula but will discuss a few of them.

This semester I am teaching Analysis I — a.k.a., Calculus Done Right — using the wonderful book by MAA President David Bressoud. *A Radical Approach to Real Analysis* fits my background and teaching style perfectly. I tell the students on the first day that this historical approach, where we repeat some of the errors that famous mathematicians have made, is harder than the modern definition-theorem-proof style that is standard today but that they will learn more, understand more, and gain a better appreciation of mathematics and how it is done. I admit that this approach is not for everyone. However, every instructor of such a course should read Bressoud's book for the historical overview that it provides.

1 Students at the Blackboard

We were only a few days into the semester when we came to Exercise 2.3.5, which is in a section about computing π:

Prove Machin's identity:

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

When I first looked at this I had no clue how to begin and realized that this was a problem better suited for class discussion than for homework. Fortunately, that famous student crutch — the back of the book — had a hint:

> Take the tangent of each side and use the formula
>
> $$\tan(x+y) = \frac{\tan\ x + \tan\ y}{1 - \tan\ x\ \tan\ y}.$$

I told the students that they were to work on this ahead of time, so when I got to class, I asked one of them if he had gotten the problem. When he responded "No," I said "OK, go to the board and simplify $\tan(2\arctan(1/5))$ by using the hint." I asked a second student the same question, got the same response, and asked her to simplify $\tan(4\arctan(1/5))$. They were initially puzzled as to how to apply the hint, for there did not seem to be the sum of two angles. It was Monday, so I reminded them that $2+2=4$, and they were off and computing. The third student was instructed to attack the original problem.

We are fortunate at West Point that there are blackboards all around the room. The third student was at the center board, the others at the sides. After some algebraic and arithmetic fumbling, the three boards looked like this:

Student 1:

$$\begin{aligned}
\tan\left(2\arctan\left(\frac{1}{5}\right)\right) &= \tan\left(\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{5}\right)\right) \\
&= \frac{\tan(\arctan(\frac{1}{5})) + \tan(\arctan(\frac{1}{5})))}{1 - \tan(\arctan(\frac{1}{5})) + \tan(\arctan(\frac{1}{5})} \\
&= \frac{1/5 + 1/5}{1 - 1/25} \\
&= \frac{5}{12}
\end{aligned}$$

Student 1 was happy with his work, but wondered where the 1/5 came from.

Student 2:

$$\begin{aligned}
\tan\left(4\arctan\left(\frac{1}{5}\right)\right) &= \tan\left(2\arctan\left(\frac{1}{5}\right) + 2\arctan\left(\frac{1}{5}\right)\right) \\
&= \frac{\tan(2\arctan(\frac{1}{5})) + \tan(2\arctan(\frac{1}{5})))}{1 - \tan(2\arctan(\frac{1}{5})) + \tan(2\arctan(\frac{1}{5})}
\end{aligned}$$

At this point I interrupted Student 2 and suggested that she ask Student 1 for his answer. Then the computation continued:

$$= \frac{5/12 + 5/12}{1 - (5/12)(5/12)} = \frac{120}{119}$$

Student 3:

Taking tangents of both sides of Machin's identity, using the angle sum formula, and using the computation of Student 2, Student 3 obtained:

$$\begin{aligned}
\tan\left(\frac{\pi}{4}\right) &= \tan\left(4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)\right) \\
&= \frac{\tan(4\arctan(\frac{1}{5})) + \tan(-\arctan(\frac{1}{239})))}{1 - \tan(4\arctan(\frac{1}{5}))\cdot\tan(-\arctan(\frac{1}{239})} \\
&= \frac{\frac{120}{119} - \frac{1}{239}}{1 - \frac{120}{119}\cdot(-\frac{1}{239})} \\
&= 1
\end{aligned}$$

Now it was time to involve the rest of the class who, up to this time, had been critical observers, helping the students at the board by encouraging them and pointing out their errors. "Do you understand the details? Is the proof correct?" They quickly responded "Yes, yes," but I countered, "Not so fast, you know $\tan(\pi/4) = 1$ and so all that has been proved is that $1 = 1$ and last term in your introduction to proof course, you learned that from $M \to 1 = 1$ you can infer nothing about the truth of M." They were a good class and immediately responded that one could start at the bottom with $1 = 1$ and reverse every step, thereby obtaining a correct proof of Machin's identity. I was pleased.

Next I asked the class how they liked this proof. Did it help them understand Machin's identity? Did they understand where Machin's identity came from? Did the proof give them insight. I knew the answers before I asked, but I wanted them to realize that this was correct, but inelegant and uninspiring mathematics. This is not the way to do mathematics.

I discussed a slightly more organized proof, but Student 1, and soon the whole class, were asking where the 1/5 came from? For them it was a *deus ex machina*, something that was unacceptable to them — and rightly so. I was unable to answer their question so it was time for some historical research.

2 Off to the Rare Book Room

So how was Machin's identity originally discovered and proved? The first publication of the result was by William Jones in his *Synopsis palmariorum matheseos: or, a New Introduction to the Mathematics*, which was published in London in 1706.[1] David Eugene Smith's *A Source Book in Mathematics* (1929), provides easy access to the passages of interest. Smith records that on page 243 of the *Palmariorum*, Jones gives James Gregory's series for the arctangent:[2]

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots .$$

While we have not followed the notation that Jones used, we remark that in this context he used the symbol π as well as the word 'periphery,' which is likely the reason that the symbol π was chosen. On this, more below. Substituting $x = 1$ yields an infinite series for π:

$$\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots .$$

From a computational point of view, this series is worthless. In his first letter to Leibniz, the *epistola prior*, Newton wrote on 24 October 1676:

> to find the length of the quadrant[al arc of which the chord is unity] to twenty decimal places, it would require about 5 000 000 000 terms of the series, for the calculation of which 1000 years would be required.[3]

[1] The *Palmariorum* of William Jones (1675–1749) drew the attention of Newton and so Jones became one of the few who had access to the unpublished work of Newton. The *Palmariorum* contains a tolerably good exposition of current mathematics, including an introduction to fluxions and series, mention of Torricelli's infinite solid, the isochronous property of the cycloid, Snell's law, and external ballistics, including the remark that "because of the *Air's Resistance*, the *Line of Projects* is not exactly *Parabolical*." This friendship produced several benefits for Jones including seeing Newton's *Analysis per quantitatum series, fluxiones ac differentias; cum enumeratione linearum tertii ordinis* (1711) through the press.

[2] James Gregory (1638–1675) is not to be confused with his less famous nephew David Gregory (1659–1708). To obtain James Gregory's series, expand $1/(1 + x^2)$ as a geometric series and then integrate term by term. This series appeared in a letter from James Gregory to John Collins, 15 February 1671. See *James Gregory Tercentenary Memorial Volume* (1939), edited by H. W. Turnbull, which is reprinted, pp. 87–91, in *Pi: A Source Book*, second edition 2000, edited by Lennart Berggren, Jonathan Borwein and Peter Borwein: we cite this volume as B^3. Note that Leibniz (1673) and Nilakantha (circa 1500) also knew this. See Ranjan Roy, "The discovery of the series formula for π by Leibniz, Gregory and Nilakantha," *Mathematics Magazine*, 63 (1990), 291–306; Reprinted B^3, 92–107. In his *Vera circuli et hyperbolae quadratura* (1667), Gregory generalizes the work of Archimedes on the *Measurement of the series* by using the geometric and harmonic means to give recursive formulas for the circumferences of the inscribed and circumscribed polygons. In this paper, the *Dictionary of Scientific Biography* (DSB) has been consulted for biographical information and further references. His nephew, David Gregory was elected to the chair of mathematics at Edinburgh a month before he took his M.A. He later became Savilian Professor of Astronomy at Oxford. In the words of Tom Whiteside in the DSB, David Gregory had a "modicum of talent, effectively lacking originality, [which] was stretched a long way." Nonetheless he produced an edition of Euclid (1703), with a magnificent frontispiece. See Rickey and Florence D. Fasanelli, "Why have a frontispiece? Examples from the Michalowicz Collection at American University," *Revista Brasileica de Historia da Matematica*, 1977 (this volume is a Festschrift for Ubiratan D'Ambrosio).

[3] *The Correspondence of Isaac Newton*, vol. 2 (1960), edited by H. W. Turnbull, p. 138–139. The bracketed comment is earlier in Newton's paragraph, but incorporated here for accuracy. When Newton wrote this, π was known but to 39 places; this was by Grienberger in 1630

If a smaller value is substituted for x, then the series does become useful computationally. Jones gives the series where $x = 1/\sqrt{3}$:

$$\pi = 2\sqrt{3} - \frac{1}{3}\frac{3\cdot 2\sqrt{3}}{9} + \frac{1}{5}\frac{2\sqrt{3}}{9} - \frac{1}{7}\frac{3\cdot 2\sqrt{3}}{9^2} + \frac{1}{9}\frac{2\sqrt{3}}{9^2} - \frac{1}{11}\frac{3\cdot 2\sqrt{3}}{9^3} + \frac{1}{13}\frac{2\sqrt{3}}{9^3} - \cdots .$$

Next Jones makes a most interesting remark:

> Theref. the (Radius is to 1/2 Periphery, or) Diameter is to the Periphery, as 1,000, &c to 3.141592653 . 5897932384 . 6264338327 . 9502884197 . 1693993751 . 0582097494 . 4592307816 . 4062862089 . 9862803482 . 5342117067. 9+ True to above a [sic] 100 Places; as Computed by the accurate and Ready Pen of the Truly Ingenious Mr. *John Machin*.

From this passage one would think that the series just before this paragraph was used for the computation, but I doubt that very much, for to obtain 100 decimal places of accuracy one would need to take $n = 206$ terms. More interestingly, if this is the way Machin did it, then Machin did not use Machin's formula! This series was first used by Halley in 1699 to compute 13 "places of decimal figures" of π and then by Abraham Sharp[4] to compute 73. Sharp's computations — the plural is used because he did the approximation five ways — were published in 1741 [[Need to check this year in OCLC.]] by William Gardiner in the third edition of *Sherwin's Mathematical Tables*, but are omitted in Hutton's 1784 edition and so are reproduced by Maseres in *Scripta logarithmica*, 99–154.[5]

Smith next reports that on page 263, Jones states that

> in the *Circle*, the *Diameter* is to the *Circumference* as 1 to
>
> $$\frac{16}{3} - \frac{4}{239} - \frac{1}{3}\frac{16}{5^3} - \frac{4}{239^3} + \frac{1}{5}\frac{16}{5^5} - \frac{4}{239^5} -, \&c. = 3.14159, \&c. = \pi \ \ldots$$

When I tried to match this up with the Machin's formula as expanded by Gregory's series, i.e.,

$$\pi = 16\left(\frac{1}{1\cdot 5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \cdots\right) - 4\left(\frac{1}{1\cdot 239} - \frac{1}{3\cdot 239^3} + \frac{1}{5\cdot 2239^5} - \cdots\right),$$

[B^3, p. 220] who was one of the last to compute π by the method of Archimedes. A few years earlier, Ludolp Van Ceulen in 1610 computed π to 35 places. His work was known to Newton [[after I wrote this, I am not so sure; he is not in the index to Corr II. Check Papers.]]

[4]See Elizabeth Connor, "Abraham Sharp, 1653–1742," *Publications of the Astronomical Society of the Pacific*, vol. 54, no. 34, 1942, pp. 237–243.

[5]These page numbers refer to the second pagination in the volume. After the printing was far along, Maseres decided to include a translation of the first three chapters of Jakob Bernoulli's *Ars conjectandi*; if that is of interest to you, first consult the 2006 English translation by Edith Dudley Sylla, who incidentally, is the great-great-granddaughter of Jared Mansfield one of the first professors of mathematics and later professor of Natural and Experimental Philosophy at West Point. Finally, I note that I used the 1761 edition of Sherwood, which is at West Point.

I had considerable difficulty, so I did what historians are supposed to do, I went to the library and looked at the *Palmariorum* itself. I immediately realized that Smith had botched it. The original reads

$$\overline{\frac{16}{5}-\frac{4}{239}}-\frac{1}{3}\overline{\frac{16}{5^3}-\frac{4}{239^3}}+\frac{1}{5}\overline{\frac{16}{5^5}-\frac{4}{239^5}}-, \&c. = 3.14159, \&c. = \pi \ .$$

where here the overbars denote parentheses. Also note that while the Smith reproduction ends with an ellipsis, the original in Jones ends with a period. Smith omits the important next sentence of Jones:

> This *Series* (among others for the same purpose, and drawn from the same Principle) I receiv'd from the Excellent Analyst, and my much Esteem'd Friend Mr. *John Machin*; and by means thereof, *Van Culen's* Number, or that in Art. 64.38. [i.e., p. 263 as cited above, but there is a typo, for 64.38 should be 65.38] may be Examin'd with all desirable Ease and Dispatch.

Sadly, the *Palmariorum* gave not a clue as to why Machin's formula was true — note that Machin's formula was never stated, but only the series based on it — so I continued my adventure in the library.

Margaret Baron's article about Jones in the DSB led me to look for an article by Jones about compound interest in volume 3 of Francis Maseres' *Scriptores logarithmici; or a Collection of Several Curious Tracts on the Nature and Construction of Logarithms ...* (1796).[6] This did not seem very promising, but I decided to look anyway. To my great surprise, Maseres was a veritible gold mine — a sourcebook of translations, reprints, and commentaries on the works of several authors.[7] Of particular interest is one entitled "A most easy and expeditious method of squaring the circle, invented by the late Mr. John Machin, Professor of Astronomy in Gresham College, London, and Secretary to the Royal Society," on pp. 155–164. No author is stated either here or in the table of contents but internal evidence makes it clear that this is Maseres exposition of Machin's own solution:

> But, about the year 1752, a friend of mine, who was much acquainted with the late learned mathematician, George Lewis Scott, Esquire,

[6] Francis Maseres (1731–1824) graduated with highest honors in classics and mathematics from Clare College, Cambridge, but became a lawyer. He served as attorney general for Quebec around the time of the American Revolution and developed a lasting interest in the affairs of Canada and the American Colonies. After returning to England he wrote in opposition to the use of negative numbers (1758). His most significant publication is the *Scriptores logarithmici* which was published in six large volumes between 1791 and 1807; we shall only refer to the third volume. He writes on and on in a prolix style, often repeating himself, providing minor variants of his point and including so much detail that the reader becomes annoyed. One work that deserves republication is his edition of John colson's translation of Maria Agnesi's *Analytical Institutions* (1802).

[7] There are very few references to Maseres containing Machin's proof, or indeed to the work of Maseres at all. The only one I know of is in W. W. Rouse Ball's *Mathematical Recreations and Essays*, eleventh edition, 1962, p. 346, but I only found this after reading Maseres.

> (who was afterwards one of the Commissioners of Excise) shewed me Mr. Machin's own investigation of it, which he had received from Mr. Scott, who had been acquainted with Mr. Machin: and, when I published my Dissertation above-mentioned in the year 1758, I printed the said investigations (with Mr. Scott's consent) in an appendix to it, as a curiosity which (though no way connected with the subject of the book) the mathematical world would probably be glad to see.[8]

The book referred to is Maseres' *Dissertation on the Use of the Negative Sign in Algebra* (1758). Maseres believes this is the first book, after Jones (1706), that contains Machin's series. When Maseres published this "it was so far from being obvious, or easy to discover, that I have been informed that the very learned and sagacious Mr. Thomas Simpson, of Woolwich Academy, (though he bestowed much pains and attention upon it,) could never find it out."[9]

Maseres next gives Machin's argument, but before we give it, we shall quote from Charles Hutton's[10] *A Treatise on Mensuration, both in Theory and Practice* (1770), for it has an excellent statement of how the method works:

> As the famous quadrature of the late Mr. John Machin, Professor of Astronomy in Gresham College, is extremely expeditious, and but little known, I shall take this opportunity of explaining it as follows.
>
> Since the chief advantage consists in taking small arcs whose tangents shall be numbers easy to manage, Mr. Machin very properly considered that, since the tangent of 45° is 1, and that, the tangent

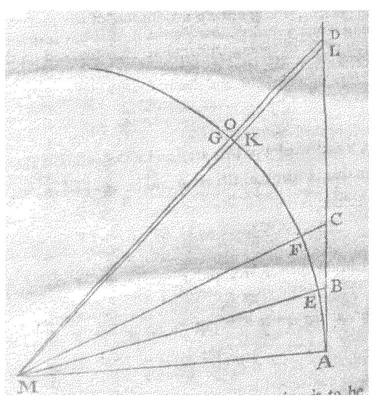
[8] P. 158 of Maseres. *** Put biographical information about Scott in this footnote.

[9] P. vii of Maseres. Thomas Simpson (1710–1761), whose name today is associated with Simpson's Rule, was known for his treatises on fluxions (1737) and probability (1742) as well as successful textbooks on algebra, geometry, and trigonometry, each of which went through numerous editions (see the DSB for details). He also edited the *Ladies' Diary*. Thomas Simpson is not to be confused with Robert Simson (1687–1768), who worked to restore "lost" works of Greek mathematics. His edition of *Euclid* (1756) influenced every later edition. On the negative side, like Maseres, he opposed negative numbers. Although Thomas Simpson never could prove Machin's formula, Robert Simson certainly did, except that he does not explain the mysterious $1/5$ either. See Ian Tweddle, "John Machin and Robert Simson on inverse-tangent series for π," *Archive for History of Exact Sciences*, 42 (1991), 1-14. I would like to thank my colleague Amanda Beecher for quickly obtaining a copy of this paper for me. Almost as an aside, Tweddle gives some information about the (at the time) unpublished history of Machin's identity: On 8 July 1706, DeMoivre wrote Johann Bernoulli informing him of Machin's series and suggesting that he inform Jacob Hermann. On 21 August 1706, Herman wrote Leibniz and reported the result $[1] = [2] + [3]$, a result which was known to Machin. Then on 6 July 1708, DeMoivre wrote Johann Bernoulli again giving two proofs that Machin's series converged to π. This point deserves further research.

[10] As a boy, Charles Hutton (1737–1823) dislocated his left elbow in a street-brawl. It did not heal properly and so he was sent to school instead of to the mines. After several years the teacher left and Hutton began a life of teaching by day and learning by night, eventually becoming professor at the Royal Military Academy in Woolwich where he remained for 43 years. He published a great deal, including *Mathematical and Philosophical Dictionary* (1795, 2 vols.), and *Recreations in Mathematics and Natural Philosophy* (1803, 4 vols), which is based on the work of Ozanam (1694) and Montucla (1778), and edited the *Ladies' Diary* for 33 years. His *A Course in Mathematics* (1898, 2 vols.) was used as a textbook for the first two decades of West Point.

of any arc being given, the tangent of double that arc can easily be had; if there be assumed some small simple number as the tangent of an arc, and then the tangent of the double arc be continually taken, until a tangent be found nearly equal to 1, which is the tangent of 45°; by taking the tangent answering to the small difference of 45° and this multiple, there would be had two very small tangents, viz. the tangent first assumed, and the tangent of the difference between 45° and the multiple arc; and that, therefore, the lengths of the arcs corresponding to these two tangents being calculated, and the arc belonging to the tangent first assumed being so often doubled as the multiple directs, the result, increased or decreased by that other arc, according as the multiple should be below or above it, would be the arc of 45°.

Having thus thought of his method, by a few trials he was lucky enough to find a number (and perhaps the only one) proper for this purpose; viz. knowing that the tangent of 1/4 of 45° is nearly = 1/5, he assumed 1/5 as the tangent of an arc.[11]



Machin begins with three lemmas that are proved in his *Elements of Plane Trigonometry*, pp. 72–80 and 424–425.

These things being premised, the method itself may be explained as follows.

3. Let AE be an arc whose tangent AB is 1/5 of the radius MA; and let AF be double, and AG quadruple, of AE, and AK an arc of 45°; and let AC, AD, AL, be the tangents of the arcs AF, AG, and AK, respectively. Put $AM = 1$, $AB = b$, $AC = c$, and $AD = d$. Then by the first of the foregoing lemmas, we shall have $c = \frac{2b}{1-bb} =$

[11] P. 165. Modern books do not seem to give any explanation for the 1/5. One that does is Tom Apostol's *Calculus*, vol. 1 (1961), p. 460. I would like to thank Bob Stein for this reference.

$\frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{\frac{2}{5}}{\frac{24}{25}} = \frac{2}{5} \times \frac{25}{24} = \frac{5}{12}$; and $d = \frac{2c}{1-cc} = \frac{\frac{10}{12}}{1-\frac{25}{144}} = \frac{\frac{10}{12}}{\frac{119}{144}} = \frac{10}{12} \times \frac{144}{119} = \frac{10\times 12}{119} = \frac{120}{119}$. Therefore d or AD, is greater than 1, or AM, and consequently than AL; and consequently AG is greater than AK, or 45°. Draw KO and tangent GK, the difference of the arcs AG, AK, (or rather, because it is so extremely small, conceive it to be drawn) and call it e; then (by lemma 2.) we shall have $e = \frac{d-1}{1+d} = \frac{\frac{120}{119}-1}{1+\frac{120}{119}} = \frac{\frac{1}{119}}{\frac{239}{119}} = \frac{1}{239}$. Find now the lengths of the arcs AE, and GK, from their tangents b and e, or $\frac{1}{5}$ and $\frac{1}{239}$, by the last of the foregoing lemmas; and from quadruple the former arc subtract the latter arc, and the remainder will be the length of an arc of 45°, which multiplied by 4 gives the length of the circumference. [p. 290–291]

3 Machin's Computation of π

Maseres now explains how the computation goes. First he computes 4 times

$$\arctan\left(\frac{1}{5}\right) = \frac{1}{5} - \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5 - \frac{1}{7}\left(\frac{1}{5}\right)^7 \cdots .$$

Since this series involves only odd powers of 1/5, we begin with $b = 1/5 = .2$ and then repeatedly multiply by $b^2 = .04$. One easily obtains:

b	= .2
b^3	= . 8,
b^5	= . ,32
b^7	= . , 12,8
b^9	= . , ,512
b^{11}	= . , , 20,48
b^{13}	= . , , ,819,2
b^{15}	= . , , , 32,768,
b^{17}	= . , , , 1,310,72
b^{19}	= . , , , , 52,428,8
b^{21}	= . , , , , 2,097,152,
b^{23}	= . , , , , , 83,886,08
b^{25}	= . , , , , , 3,355,44
b^{27}	= . , , , , , ,134,21
b^{29}	= . , , , , , , 5,36
b^{31}	= . , , , , , , ,21

Now he computes "the affirmative terms of the series":[12]

[12] The antepenultimate and penultimate entries in the table have been truncated, not rounded. This is true also of some entries below.

$$
\begin{array}{ll}
b & = .2 \\
b^5/5 & = .\quad ,\ 64, \\
b^9/9 & = .\quad ,\quad ,\ 56,888,888,888,888,8 \\
b^{13}/13 & = .\quad ,\quad ,\quad ,\ 63,015,384,615,3 \\
b^{17}/17 & = .\quad ,\quad ,\quad ,\quad ,\ 77,101,176,4 \\
b^{21}/21 & = .\quad ,\quad ,\quad ,\quad ,\quad ,\ 99,864,3 \\
b^{25}/25 & = .\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,134,2 \\
b^{29}/29 & = .\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,1
\end{array}
$$

The sum of these positive terms is $.200,064,056,951,981,474,679,1$. Next, he computes the negative terms from $b^3/3$ to $b^{31}/31$ and finds that their sum is $.002,668,497,102,100,716,309,1$. Subtracting and multiplying by 4 yields $4\arctan(1/5) = .789,582,239,399,523,033,480,0 = AG$.

Now we need to deal with $\arctan(1/239)$. To compute odd powers of $e = 1/239$, we repeatedly divide by $239^2 = 57121$. This yields:

$$
\begin{array}{ll}
e & = .004,184,100,418,410,041,841,0 \\
e^3 & = .\quad ,\quad ,\ 73,249,775,361,251,4 \\
e^5 & = .\quad ,\quad ,\quad ,\ 1,282,261,572,1 \\
e^7 & = .\quad ,\quad ,\quad ,\quad ,\quad ,\ 22,449,9 \\
e^9 & = .\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,3
\end{array}
$$

First the positive terms:

$$
\begin{array}{ll}
e & = .004,184,100,418,410,041,841,0 \\
e^5/5 & = .\quad ,\quad ,\quad ,\quad ,256,472,314,4 \\
e^9/9 & = .\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,0
\end{array}
$$

And now the negative terms:

$$
\begin{array}{ll}
e^3/3 & = .000,000,024,416,591,787,083,8 \\
e^7/7 & = .\quad ,\quad ,\quad ,\quad ,\quad ,\ 3,207,1
\end{array}
$$

Subtracting the sum of the negative terms from the sum of the positive terms yields $\arctan(1/239) = .004,182,076,002,723,864,5 = GK$. Subtracting this from the value of AG we obtain $.785,398,163,397,448,309,615,5$, the length of "an arch of 45°." Finally, we multiply by 4 to obtain

$$
\pi = 3.141,592,653,589,793,238,462,0
$$

"the length of the semicircumference of a circle whose radius is = 1; which is true to the 21st place of decimal figures, the error being in the 22nd or last figure, which ought to be a 6 instead of a cypher."

Just to compute π "true to the 21st place of decimal figures" has required considerable labor. One can hardly imagine how much more was required to compute 100. Who of us today would have the accuracy or tenacity to do it.

4 An Indoor Sport

Machin's formula inspired many to devise similar arctangent formulas for π that would provide ease of computation and quicker convergence. From here on, to avoid fractions we use arccotangents instead of arctangents, and to simplify the typography further we shall write $\arctan(x)$ as $[x]$. Thus, for example, Machin's formula becomes simply $[1] = [5] + [239]$.

In his *Introductio in analysin infinitorum* (1748), §142, Euler gave the formula $[1] = [2]+[3]$, but did not compute an approximation of π. A decade earlier, Euler wrote a paper whose title translates as "On various ways of closely approximating numbers for the quadrature of the circle." [E74][13] In this paper, he states what we will express as

$$[1] = [a] + [b] \quad \text{where} \quad ab - 1 = a + b.$$

Results such as this reduce the question of finding Machin type identities to the solution of Diophantine equations. On this topic, there is a large literature that we shall not delve into.[14]

5 The symbol π

History of mathematics books today report that William Jones introduced the symbol π for the ratio of the circumference to the diameter of a circle in his *Synopsis palmariorum matheseos* of 1706. But it was not always so. Benjamin Franklin Finkel, founding editor of the *American Mathematical Monthly*, writing in 1897, presented the opinion that had been widely held since 1748:

> The use of a single symbol to denote 3.14159265 … appears to have been introduced by John Bernoulli, who represented it by the letter c. Euler in 1734 denoted it by p, and in a letter of 1736 in which he enunciated the theorem that the sum of the square of the reciprocals of the natural numbers is $\frac{1}{6}\pi^2$, he uses the letter c. Chr. Goldbach in 1742 used π, and after the publication of Euler's *Analysis* the

[13] Euler affadacios refer to this paper as E74, using the catalog number that Gustav Eneström assigned to it. Essentially all of Euler's papers are available in their original format, and more and more in English translation, on the Euler Archive (`http://www.math.dartmouth.edu/~euler/`). For those of you who do not read Latin (it is not as hard as you may think), see the discussion of this paper in C. Edward Sandifer, *The Early Mathematics of Leonhard Euler* (2007), pp. 274–278.

[14] For details, see J. W. Wrench, Jr., "On the derivation of arctangent equalities," *The American Mathematical Monthly*, 45 (1938), 108–109. D. H. Lehmer, "On arctangent relations for π," *AMM* 45 (1938), 657–664, introduced the square bracket notation, a measure for how computationally efficient the various identities are, and the phrase "indoor sport" to refer to the search for Machin type identities. For a more recent entry into the literature, see Michael Wetherfield, "The enhancement of Machin's formula by Todd's process," *The Mathematical Gazette*, vol. 80, no. 488, July 1996, pp. 333–344.

> symbol π was generally employed, the choice of π being determined by the initial of the word $\pi\varepsilon\rho\iota\phi\varepsilon'\rho\varepsilon\iota\alpha$ — periphereia.[15]

In 1748 in his *Introductio in analysin infinitorum* [E101], Euler writes:

> Ponamus ergo Radium Circuli seu Sinum totum esse = 1, atque satis liquet Peripheriam hujus Circuli in numeris rationalibus exacte exprimi non posse, per approximationes autem inventa est Semicircumferentia hujus Circuli esse = 3,14159 [etc to 127 decimal places], pro quo numero, brevitatis ergo, *scribam* [italics added] π, ita ut sit π = Semicircumferentiae Circuli, cujus Radius = 1, seu π erit longitudo Arcus 180 graduum. [E101, §126, p. 93][16]

This has been translated by John D. Blanton in *Introduction to Analysis of the Infinite* (1988) as

> We let the radius, or total sine, of a circle to be equal to 1, then it is clear enough that the circumference of the circle cannot be expressed exactly as a rational number. An approximation of half the circumference of this circle is 3.14159 [etc to 127 decimal places]. For the sake of brevity *we will use* [italics added] the symbol π for this number. We say, then, that half the circumference of a unit circle is π, or that the length of an arc of 180 degrees is π. [§126, p. 102]

There is a great deal to be said about this passage,[17] but I only want to comment on the phrase "we will use," which has been mistranslated. Euler uses the word "scribam," which is first person singular, "I write." This means that Euler, himself, thought that he was introducing the symbol "π" in its modern sense.[18]

[15] B. F. Finkel, "Leonhard Euler," *The American Mathematical Monthly*, 4 (1897), 297–302. Reprinted pp. 5–12, in *The Genius of Euler*, edited by William Dunham, MAA 2007. Euler first used they symbol π denoting the ratio of the circumference to diameter of a circle in print in his *Mechanica* of 1736 (E15, §294). Euler used it again in print in 1741 (E63) when he wrote "Il est clair que j'emplois la lettre π pour marquer le nombre de Ludolf a Keulen 3,14159265 etc." Note the usage of the first person singular.

[16] The value of π given here by Euler is due to Thomas Fantet de Lagny (1660–1734), "Mémoire surl la quadrature du cercle, & sur la mesure de tout arc, tout secteur & tout segment donné," *M'emoires de l'Académie des sciences de Paris ????* (for 1719, published 1721), pp. 135–145. An error in this value was found by Jurij Vega (1754–1802); see his *Thesaurus logarithmorum completus* (1794),p. 633. Vega, a Professor of Mathematics at the Artillery School in Vienna, was murdered for his watch. See Edward Sandifer, "Why 140 digits of pi matter," http://www.southernct.edu/ sandifer/Ed/History/Preprints/Talks/Jurij%20Vega/Vega%20math%20script.pdf).

[17] Please allow a few more comments: (1) The phrase "sinus totus" is used here for essentially the last time, as Euler's book was so influential that the unit circle (to use the modern phrase that Blanton uses) was used thereafter. (2) Euler was not the first to use a radius of 1. (3) Euler states that π is irrational. This is an instance of he truth/proof issue; Euler knows that π is irrational, but is unable to prove it.

[18] Dirk J. Struik in *A Source Book in Mathematics, 1200–1800* (1969), p. 347 correctly has "I." So does the 1796 French translation of the *Introductio* by J. B. Labey.

Maseres, p. 196, was probably the first to weigh in on who was first to introduce the symbol π. "It is possible, however, that Mr. Euler may likewise have conceived the same idea of applying the said proportion to the investigation of the same quantity by dint of his own sagacity, and without having borrowed from Mr. Machin." Indeed, "it is no means certain that he had seen even the very short and obscure description of Mr. Machin's quadrature" in Jones. Euler "may therefore very reasonably be supposed to have been *an inventor* of this excellent method of performing the quadrature of the circle as well as Mr. Machin; but Mr. Machin was certainly *the first inventor* of it." By contrast, Struik comes down on the side of Jones: "Euler adopted it and provided for its universal acceptance through his *Introductio in analysin infinitorum* (Lausanne, 1748)" [*Source Book*, p. 249].

In E705, Euler mentions that Machin computed π to 100 places, but does not mention the *Palmariorum*. To find out how Euler learned of Machin and whether he ever saw a copy of Jones, I need to find a copy of F. C. Mayer's *Arithmetic of Sines* (1727), which, according to W. W. Rouse Ball, *Short Account*, Fourth edition, is the basis of Chapter VIII of Euler's *Introductio*, the chapter on trigonometry. Perhaps this will reveal how Euler learned of the work of Machin.

6 Who was John Machin?

Since Machin's name is little known today, one might assume, from his computations above, that he was just an *idiot savant*, but he was, in fact, an accomplished mathematician.

Before Brook Taylor went to Cambridge, Machin tutored him. They became friends and in a letter of 26 July 1712, Taylor wrote to Machin announcing what we call Taylor's Theorem, even noting that Machin had given him the idea in a coffeehouse conversation.[19]

Machin's work on π attracted the attention of Newton and he was chosen to be one of the members of the committee, with Halley and Jones, to prepare the *Commercium epistolicum* (1712) for the press. This was the Royal Society's "investigation" into the Newton-Leibniz controversy; hypocritically, the report was written by Newton. Later Machin edited the second edition of Newton's *Arithmetica universalis* (1722).

On 30 November 1710 Machin was elected a Fellow of the Royal Society, and then on 16 May 1713 Gresham Professor of Astronomy. From 1718 to 1747 he served as secretary of the Royal Society of London.

Lunar theory was Machin's primary interest. Indeed, a proposition of his dealing with the nodes of the moon is included in the third edition of the *Principia*.[20] Newton introduces him as "Mr. Machin, Gresham Professor" but in

[19] The letter is reproduced in *Bibliotheca Mathematica* (3) 7, (1906–1907), pp. 368-371.

[20] *Isaac Newton. The Principia. Mathematical Principles of Natural Philosophy* (1999), translated by I. Bernard Cohen and Anne Whitman, pp. 860–864.

an 1803 edition of the *Principia* edited by Davis, this was mistranslated as "Professor Gresham."[21]

John Conduitt (1688–1737), Newton's nephew-in-law, collected biographical information about Newton after he died in 1727. He noted that "Sr. I. told me that Machin understood his *Principia* better than anyone, that Halley was the best Astronomer [but] Machin the best Geometer."

7 One More Thing

This concludes our research. But there is one thing still to do. I must tell my students about what I have found and show them that the mysterious 1/5 is not so mysterious after all. For this I will wait till later in the term when we make a visit to the Rare Book Room at West Point. I intend to show and discuss books which are important in the history of analysis, including all those that have been used above.[22]

---

[21] Derek Gjersten, *The Newton Handbook* (1986), p. 331.

[22] For a list of what was shown on a previous visit see http://www.dean.usma.edu/departments/math/people/rickey/hm/09-05-01-RareBooksForAnalysis.html . For detailed information about the mathematical treasures in the West Point library, see Joe Albree, David C. Arney, and V. Frederick Rickey, *A Station Favorable to the Pursuits of Science: Primary Materials in the History of Mathematics at the United States Military Library* (2000).