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FINITE COMBINATORY PROCESSES—FORMULATION 1

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The present formulation should prove significant in the development of symbolic logic along the lines of Gödel's theorem on the incompleteness of symbolic logics[1] and Church's results concerning absolutely unsolvable problems.[2]

We have in mind a *general problem* consisting of a class of *specific problems*. A solution of the general problem will then be one which furnishes an answer to each specific problem.

In the following formulation of such a solution two concepts are involved: that of a *symbol space* in which the work leading from problem to answer is to be carried out,[3] and a fixed unalterable *set of directions* which will both direct operations in the symbol space and determine the order in which those directions are to be applied.

In the present formulation the symbol space is to consist of a two way infinite sequence of spaces or boxes, i.e., ordinally similar to the series of integers $\cdots$, $-3, -2, -1, 0, 1, 2, 3, \cdots$. The problem solver or worker is to move and work in this symbol space, being capable of being in, and operating in but one box at a time. And apart from the presence of the worker, a box is to admit of but two possible conditions, i.e., being empty or unmarked, and having a single mark in it, say a vertical stroke.

One box is to be singled out and called the starting point. We now further assume that a specific problem is to be given in symbolic form by a finite number of boxes being marked with a stroke. Likewise the answer is to be given in symbolic form by such a configuration of marked boxes. To be specific, the answer is to be the configuration of marked boxes left at the conclusion of the solving process.

The worker is assumed to be capable of performing the following primitive acts:[4]

(a) *Marking the box he is in* (*assumed empty*),
(b) *Erasing the mark in the box he is in* (*assumed marked*),
(c) *Moving to the box on his right,*
(d) *Moving to the box on his left,*
(e) *Determining whether the box he is in, is or is not marked.*

The set of directions which, be it noted, is the same for all specific problems and thus corresponds to the general problem, is to be of the following form. It is to be headed:

Start at the starting point and follow direction 1.

Received October 7, 1936. The reader should compare an article by A. M. Turing, *On computable numbers*, shortly forthcoming in the **Proceedings of the London Mathematical Society**. The present article, however, although bearing a later date, was written entirely independently of Turing's. *Editor*.

[1] Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, **Monatshefte für Mathematik und Physik**, vol. 38 (1931), pp. 173–198.

[2] Alonzo Church, *An unsolvable problem of elementary number theory*, **American Journal of Mathematics**, vol. 58 (1936), pp. 345–363.

[3] Symbol space, and time.

[4] As well as otherwise following the directions described below.

It is then to consist of a finite number of directions to be numbered 1, 2, 3, $\cdots n$. The ith direction is then to have one of the following forms:

(A) *Perform operation* $O_i[O_i=$ (a), (b), (c), *or* (d)$]$ *and then follow direction* j_i,

(B) *Perform operation* (e) *and according as the answer is yes or no correspondingly follow direction* j_i' *or* j_i'',

(C) *Stop.*

Clearly but one direction need be of type C. Note also that the state of the symbol space directly affects the process only through directions of type B.

A set of directions will be said to be *applicable* to a given general problem if in its application to each specific problem it never orders operation (a) when the box the worker is in is marked, or (b) when it is unmarked.[5] A set of directions applicable to a general problem sets up a deterministic process when applied to each specific problem. This process will terminate when and only when it comes to the direction of type (C). The set of directions will then be said to set up a *finite 1-process* in connection with the general problem if it is applicable to the problem and *if the process it determines terminates for each specific problem.* A finite 1-process associated with a general problem will be said to be a *1-solution* of the problem if the answer it thus yields for each specific problem is always correct.

We do not concern ourselves here with how the configuration of marked boxes corresponding to a specific problem, and that corresponding to its answer, symbolize the meaningful problem and answer. In fact the above assumes the specific problem to be given in symbolized form by an outside agency and, presumably, the symbolic answer likewise to be received. A more self-contained development ensues as follows. The general problem clearly consists of at most an enumerable infinity of specific problems. We need not consider the finite case. Imagine then a one-to-one correspondence set up between the class of positive integers and the class of specific problems. We can, rather arbitrarily, represent the positive integer n by marking the first n boxes to the right of the starting point. The general problem will then be said to be *1-given* if a finite 1-process is set up which, when applied to the class of positive integers as thus symbolized, yields in one-to-one fashion the class of specific problems constituting the general problem. It is convenient further to assume that when the general problem is thus 1-given each specific process at its termination leaves the worker at the starting point. If then a general problem is 1-given and 1-solved, with some obvious changes we can combine the two sets of directions to yield a finite 1-process which gives the answer to each specific problem when the latter is merely given by its number in symbolic form.

With some modification the above formulation is also applicable to symbolic logics. We do not now have a class of specific problems but a single initial finite marking of the symbol space to symbolize the primitive formal assertions of the logic. On the other hand, there will now be no direction of type (C). Consequently, assuming applicability, a deterministic process will be set up which is *unending*. We further assume that in the course of this process certain recognizable symbol groups, i.e., finite sequences of marked and unmarked boxes, will appear which are not further altered in the course of the process. These will be the derived assertions of the logic. Of course the set of directions corresponds to the deductive processes of the logic. The logic may then be said to be *1-generated*.

An alternative procedure, less in keeping, however, with the spirit of symbolic

[5] While our formulation of the set of directions could easily have been so framed that applicability would immediately be assured it seems undesirable to do so for a variety of reasons.

logic, would be to set up a finite 1-process which would yield the nth theorem or formal assertion of the logic given n, again symbolized as above.

Our initial concept of a given specific problem involves a difficulty which should be mentioned. To wit, if an outside agency gives the initial finite marking of the symbol space there is no way for us to determine, for example, which is the first and which the last marked box. This difficulty is completely avoided when the general problem is 1-given. It has also been successfully avoided whenever a finite 1-process has been set up. In practice the meaningful specific problems would be so symbolized that the bounds of such a symbolization would be recognizable by characteristic groups of marked and unmarked boxes.

The root of our difficulty however, probably lies in our assumption of an infinite symbol space. In the present formulation the boxes are, conceptually at least, physical entities, e.g., contiguous squares. Our outside agency could no more give us an infinite number of these boxes than he could mark an infinity of them assumed given. If then he presents us with the specific problem in a finite strip of such a symbol space the difficulty vanishes. Of course this would require an extension of the primitive operations to allow for the necessary extension of the given finite symbol space as the process proceeds. A final version of a formulation of the present type would therefore also set up directions for generating the symbol space.[6]

The writer expects the present formulation to turn out to be logically equivalent to recursiveness in the sense of the Gödel-Church development.[7] Its purpose, however, is not only to present a system of a certain logical potency but also, in its restricted field, of psychological fidelity. In the latter sense wider and wider formulations are contemplated. On the other hand, our aim will be to show that all such are logically reducible to formulation 1. We offer this conclusion at the present moment as a *working hypothesis*. And to our mind such is Church's identification of effective calculability with recursiveness.[8] Out of this hypothesis, and because of its apparent contradiction to all mathematical development starting with Cantor's proof of the non-enumerability of the points of a line, independently flows a Gödel-Church development. The success of the above program would, for us, change this hypothesis not so much to a definition or to an axiom but to a *natural law*. Only so, it seems to the writer, can Gödel's theorem concerning the incompleteness of symbolic logics of a certain general type and Church's results on the recursive unsolvability of certain problems be transformed into conclusions concerning all symbolic logics and all methods of solvability.

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[6] The development of formulation 1 tends in its initial stages to be rather tricky. As this is not in keeping with the spirit of such a formulation the definitive form of this formulation may relinquish some of its present simplicity to achieve greater flexibility. Having more than one way of marking a box is one possibility. The desired naturalness of development may perhaps better be achieved by allowing a finite number, perhaps two, of physical objects to serve as pointers, which the worker can identify and move from box to box.

[7] The comparison can perhaps most easily be made by defining a 1-function and proving the definition equivalent to that of recursive function. (See Church, loc. cit., p. 350.) A 1-function $f(n)$ in the field of positive integers would be one for which a finite 1-process can be set up which for each positive integer n as problem would yield $f(n)$ as answer, n and $f(n)$ symbolized as above.

[8] Cf. Church, loc. cit., pp. 346, 356–358. Actually the work already done by Church and others carries this identification considerably beyond the working hypothesis stage. But to mask this identification under a definition hides the fact that a fundamental discovery in the limitations of the mathematicizing power of Homo Sapiens has been made and blinds us to the need of its continual verification.