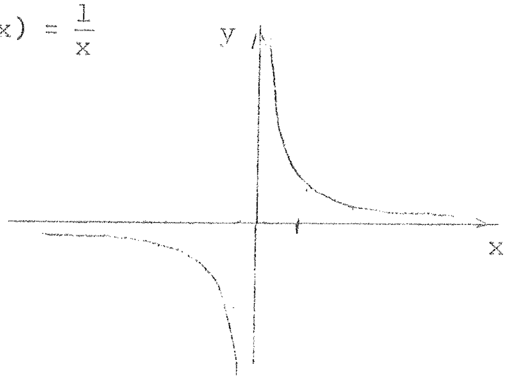
RATIONAL FUNCTIONS

A function f is a <u>rational</u> <u>function</u> if it is a quotient of polynomials, i.e., if $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials. As before we shall always assume that p and q have been completely factored into linear factors. Moreover, we assume that p and q have no common factors.

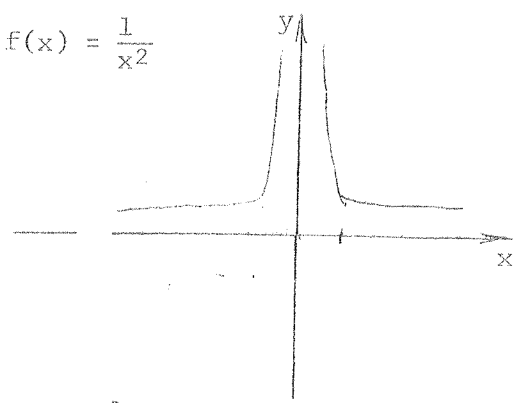
The basic building blocks for rational functions are the reciprocals of powers, i.e., the functions $f(x) = \frac{1}{x^n}$, where n is a positive integer. We begin by graphing these functions:

BASIC GRAPHS II.

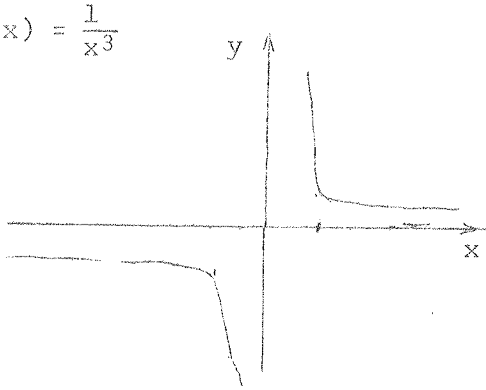
$f(x) = \frac{1}{x}$



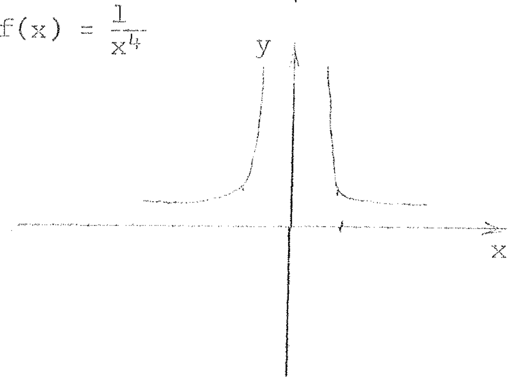
$f(x) = \frac{1}{x^2}$



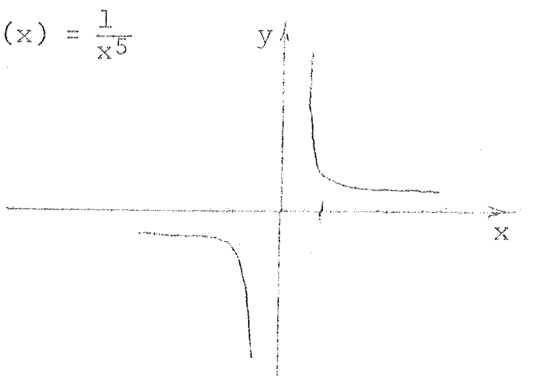
$f(x) = \frac{1}{x^3}$



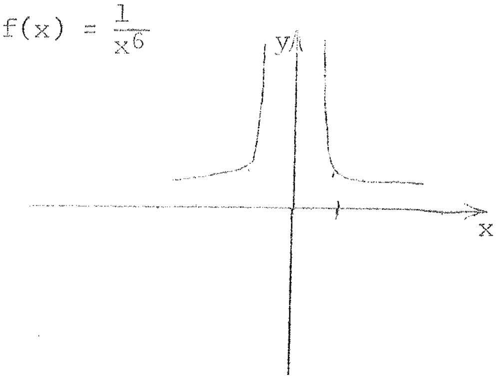
$f(x) = \frac{1}{x^4}$



$f(x) = \frac{1}{x^5}$



$f(x) = \frac{1}{x^6}$



There are several important things to be observed about these graphs.

1) They all go through the point $(1,1)$.

2) Where n is even the graphs go through $(-1,1)$, while if n is odd they go through $(-1,-1)$.

3) The graph stays on the same side of the x-axis if n is even and jumps across the axis if n is odd.

4) There is no point on the y-axis. This is because $f(0)$ is undefined (as division by zero is undefined).

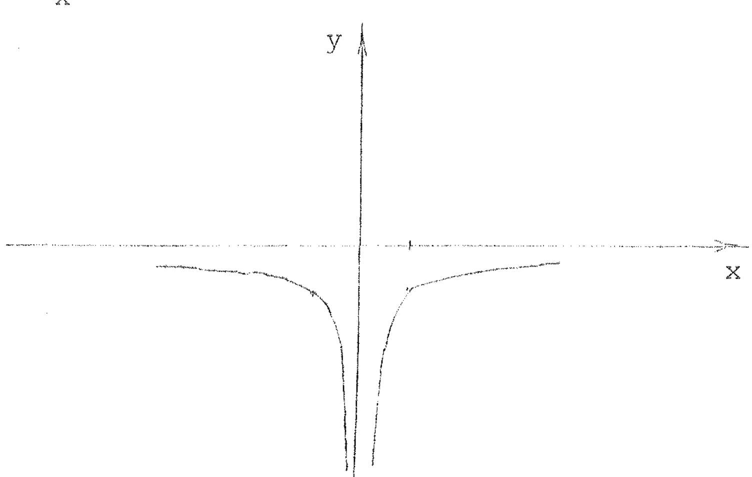
5) For x near 0, $\frac{1}{x^n}$ is large. More precisely, as x gets small, $\frac{1}{x^n}$ gets very large. To describe this behavior we say that the y-axis is a vertical asymptote and that 0 is a pole of the function.

6) For x large (either positively or negatively), $\frac{1}{x^n}$ is small. In fact, as x gets large, $\frac{1}{x^n}$ gets smaller and smaller. We say that the x-axis is a horizontal asymptote of the curve.

Now we introduce two variations of these basic graphs.

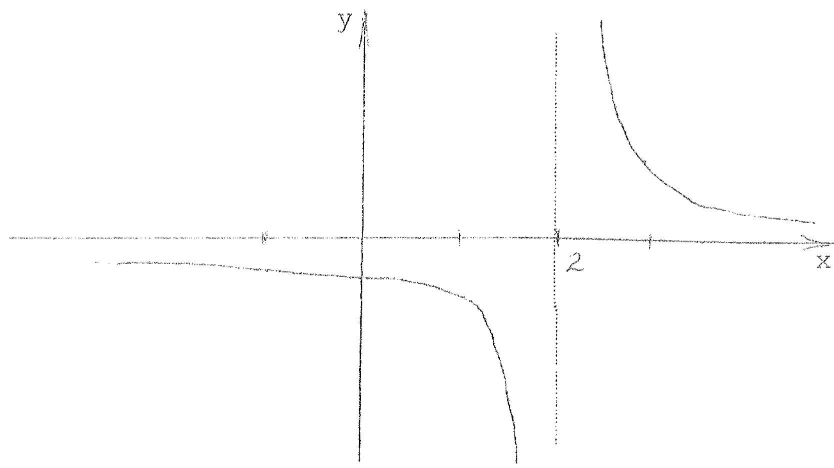
Example 22. $f(x) = -\frac{1}{x^2}$

This looks like $\frac{1}{x^2}$ except for the minus sign, so the graph looks like the graph of $f(x) = \frac{1}{x^2}$ "flipped over:"



Example 23. $f(x) = \frac{1}{(x-2)^3}$

Again this resembles the graph of $f(x) = \frac{1}{x^3}$ except that the denominator is zero at 2 instead of at 0. Thus the line $x = 2$ is a vertical asymptote; otherwise said, 2 is a pole. The graph is



Note that the vertical asymptote (when it is not the y-axis) is marked with a dotted line.

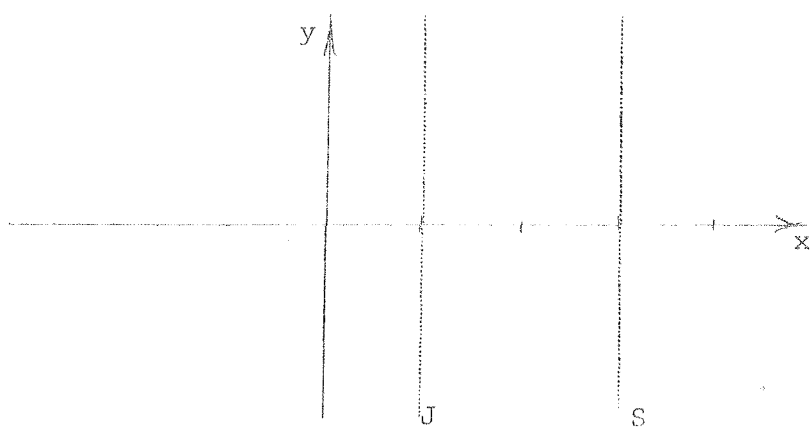
Now we come to the MAIN METHOD:

Suppose we have a rational function, the denominator and numerator of which have been completely factored into linear factors. If there is a factor $(x - a)^n$ in the denominator, then there is a pole of order n at a. Thus the graph resembles that of $y = \pm \frac{1}{x^n}$ near $x = a$. If n is odd the graph jumps across the x-axis, while if n is even it stays on the same side of the x-axis. If there is a factor $(x - b)^m$ in the numerator then b is a zero of multiplicity m. The graph looks like that of $y = \pm x^m$ at a. The graph should not have any breaks except where the curve goes from the right side to the left side of a vertical asymptote (continuity).

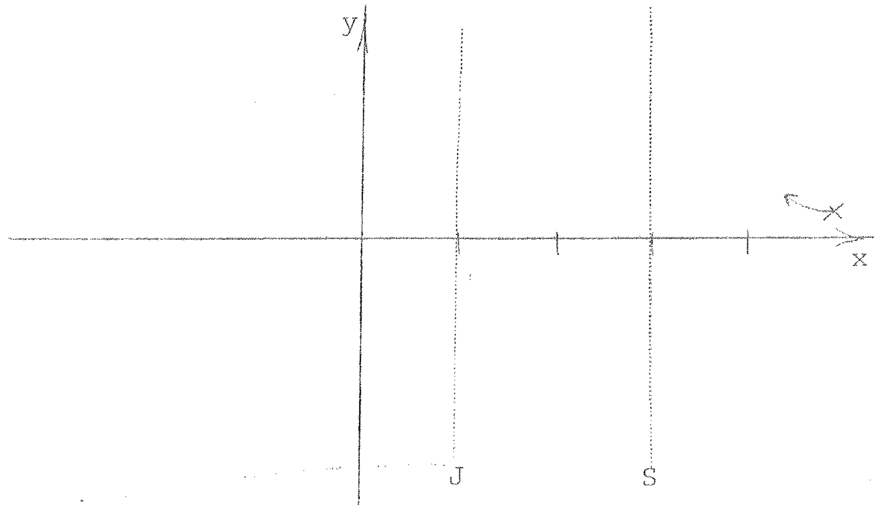
Note that in the following examples each vertical asymptote is indicated with a dotted line and each is marked J or S according as the graph "jumps" or "stays."

<u>Example 24</u>. $f(x) = \dfrac{1}{(x - 1)(x - 3)^2}$

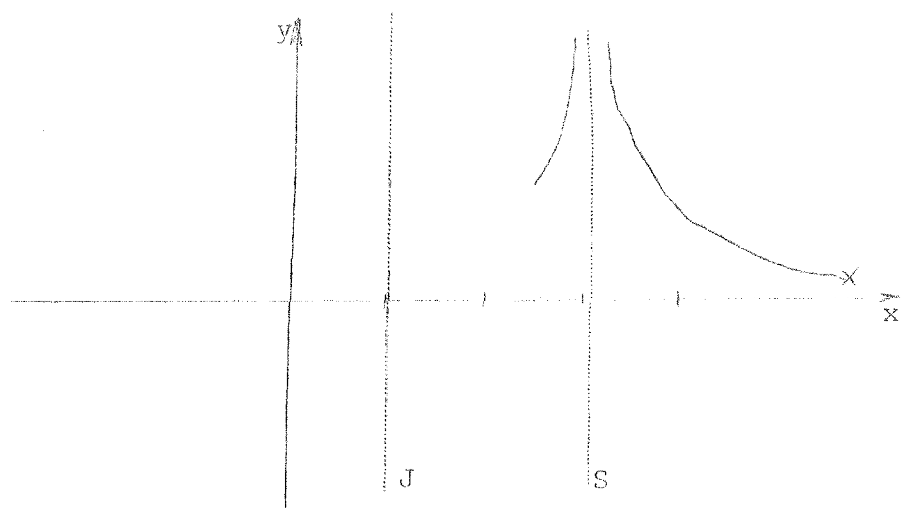
This function has two poles, as the denominator is zero at 1 and 3. The lines $x = 1$ and $x = 3$ are vertical asymptotes. As $x - 1$ occurs to an odd power, the asymptote $x = 1$ is of the "jump" type. As $x - 3$ occurs to an even power, the asymptote $x - 2$ is of the "stay" type. We now indicate the asymptotes with dotted lines in the diagram and indicate below the asymptotes whether they are of the stay or jump type:



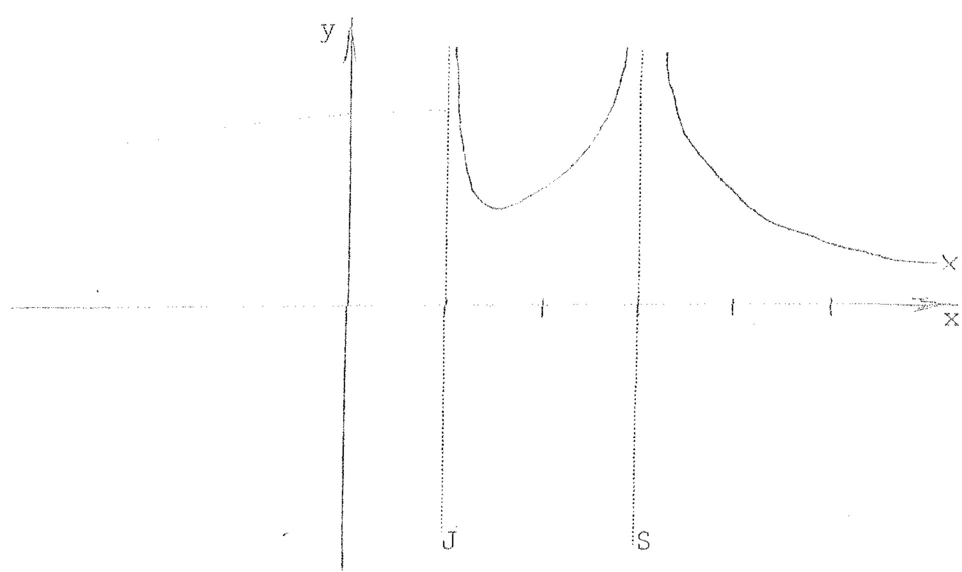
Next we need to find a "starting point." If x is large and positive, then the value of $f(x)$ is positive, but very small. So the right "end" of the graph is very close to the x-axis. We mark this point with a cross:



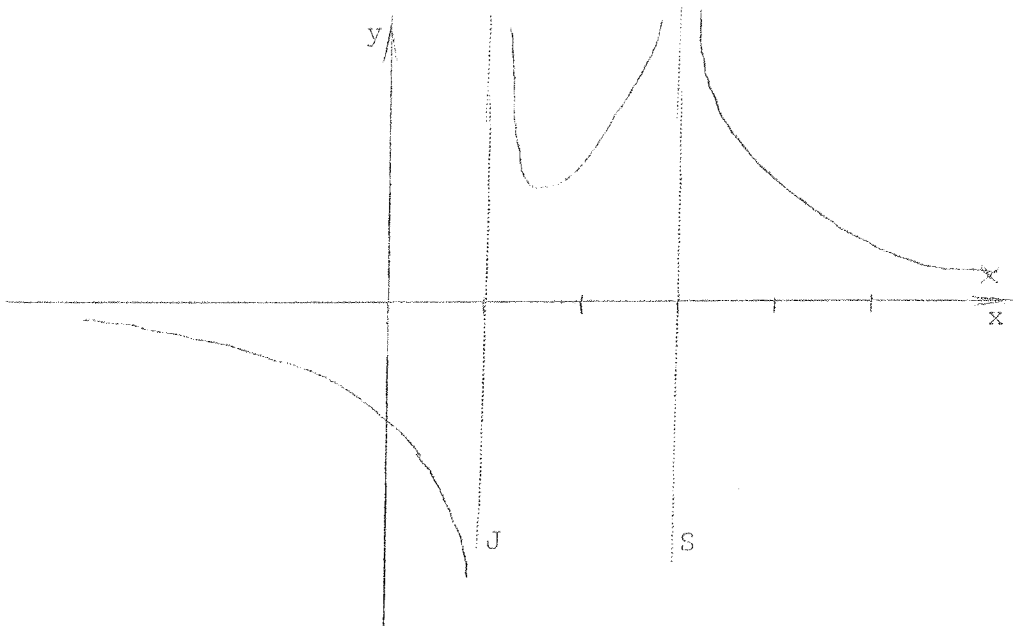
As the graph must look like that of $y = \pm \frac{1}{x^2}$ near 3 we get:



The function cannot intersect the x-axis as it has no zeros. Thus the graph must turn around and go back up again.

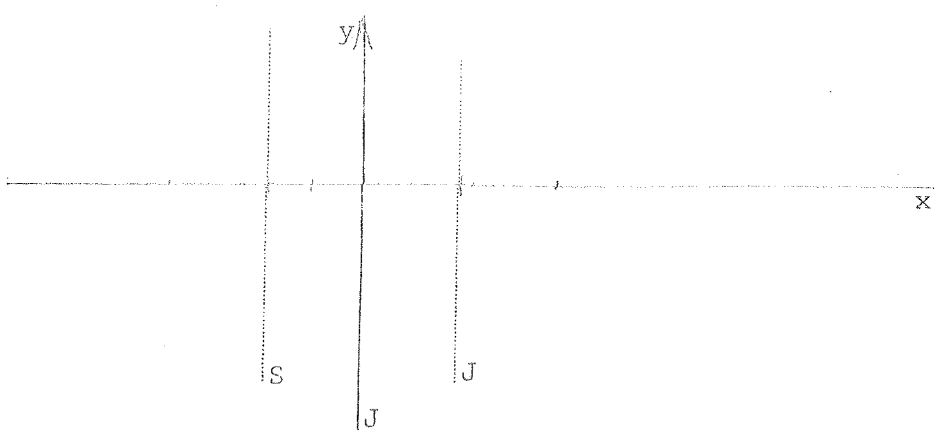


Finally, it must look like $y = \frac{1}{x}$ near 1 so we finally obtain:

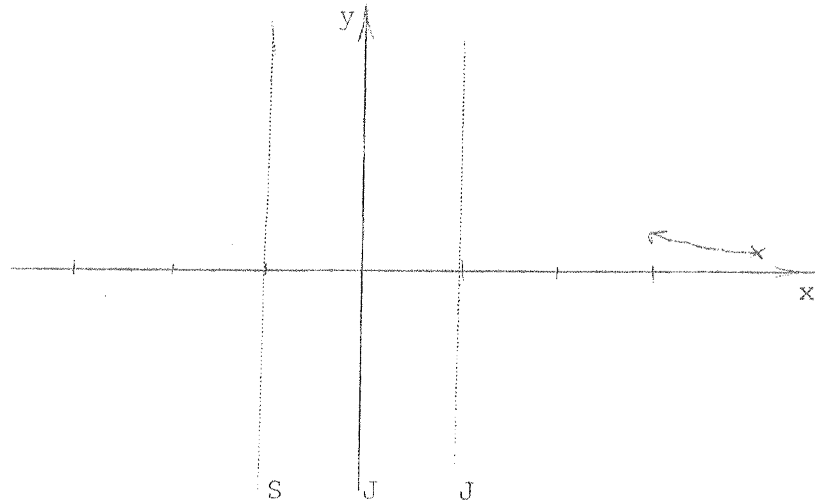


<u>Example 25</u>. Graph $f(x) = \dfrac{1}{x(x - 1)^3(x + 1)^2}$

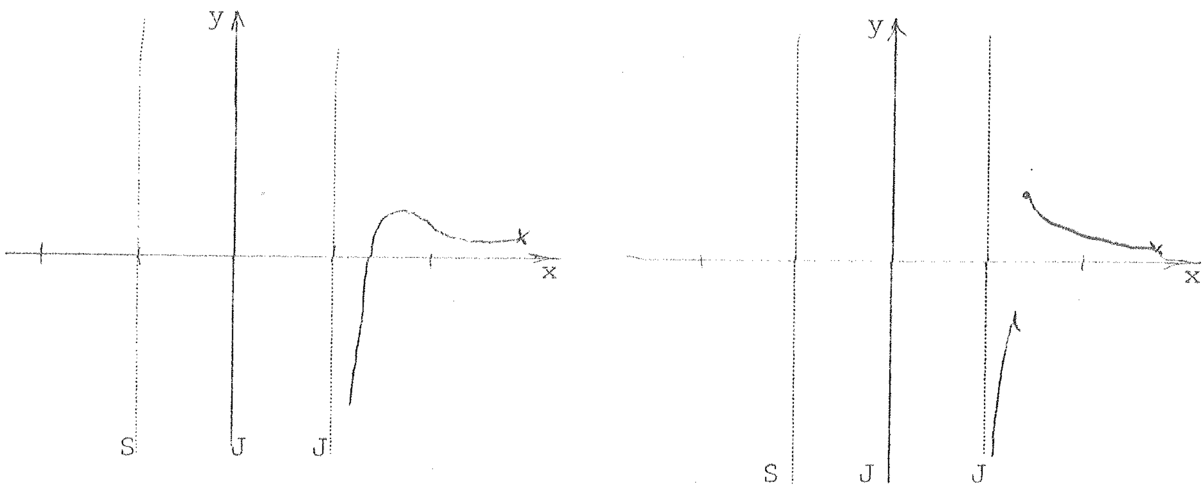
There are three vertical asymptotes. They are the lines $x = 0$, $x = 1$ and $x = -1$. Again we indicate whether the curve jumps across the x-axis as we cross the asymptote or stays on the same side of the x-axis.



If x is large and positive, $f(x)$ is positive, but very small. So the x-axis is a horizontal asymptote. The starting point is marked with a cross and we begin to draw the graph from right to left:

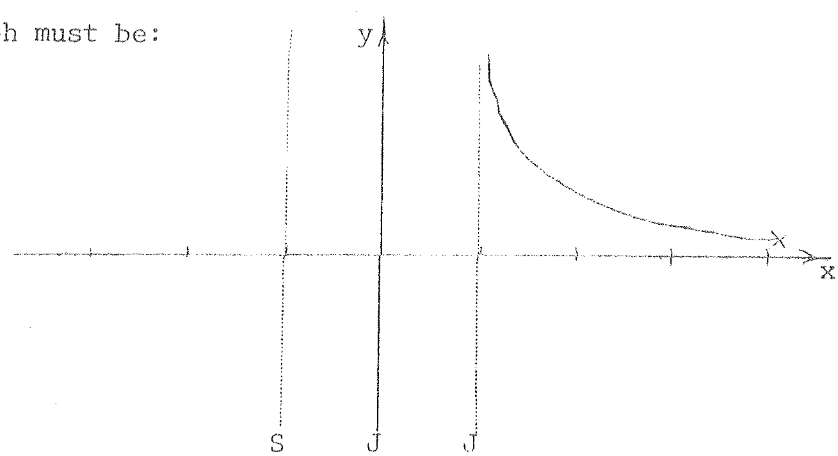


Now the graph must go up along the right side of the asymptote $x = 1$. It cannot do either of the following:

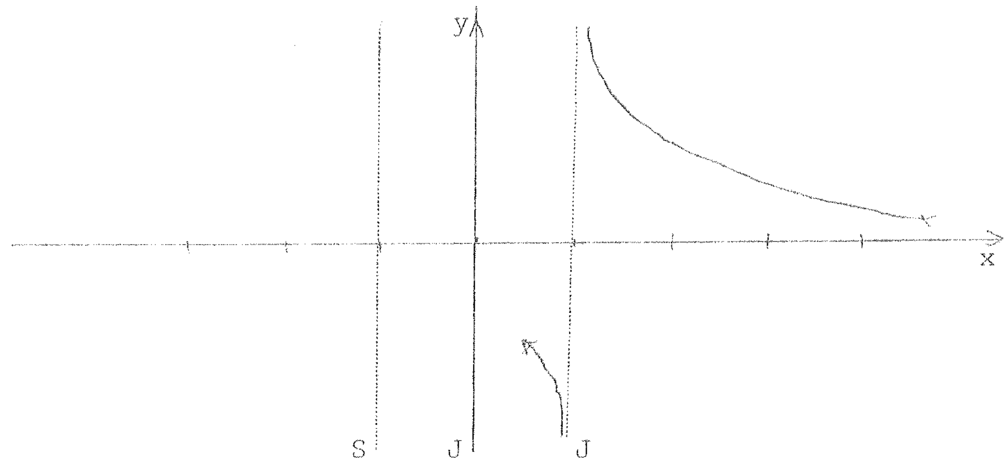


The first of these is ruled out as this function has no zeroes and thus the graph cannot intersect the x-axis (remember a fraction is zero if and only if the numerator is zero). The second cannot occur as the graph cannot have any breaks (except where it crosses a vertical asymptote).

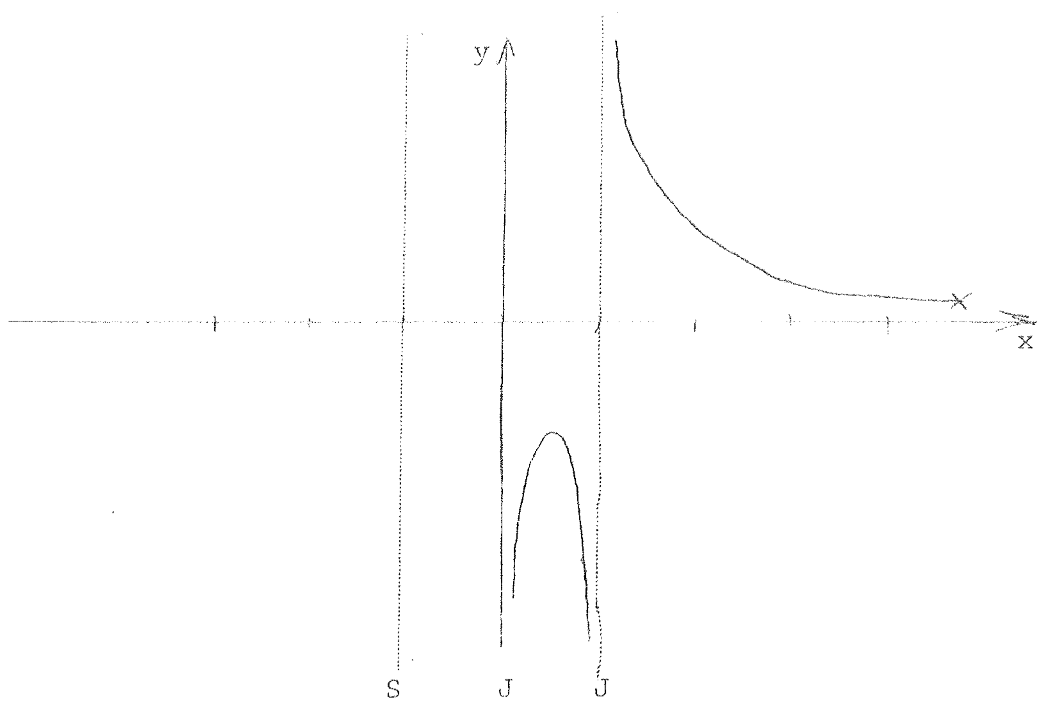
The graph must be:



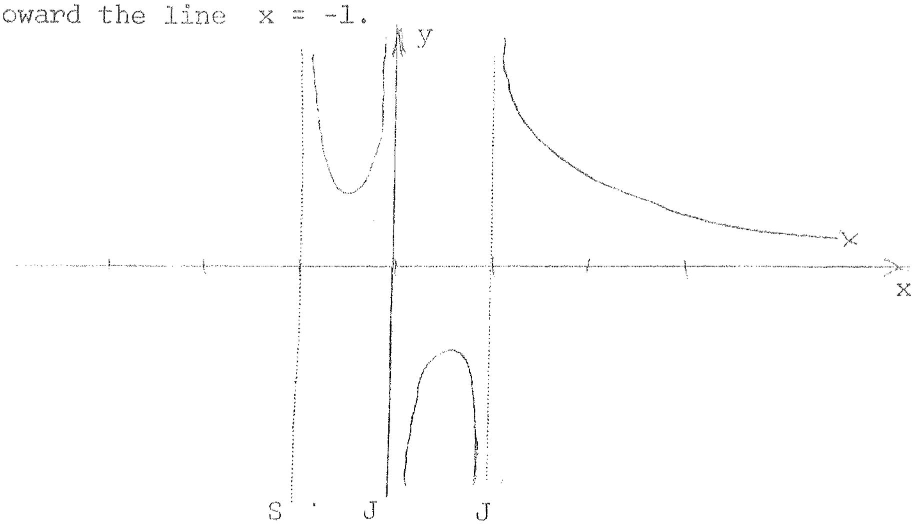
As the graph must look like that of $y = \pm \frac{1}{x^3}$ near 1, the graph jumps below the axis and comes up along the other side of the asymptote:



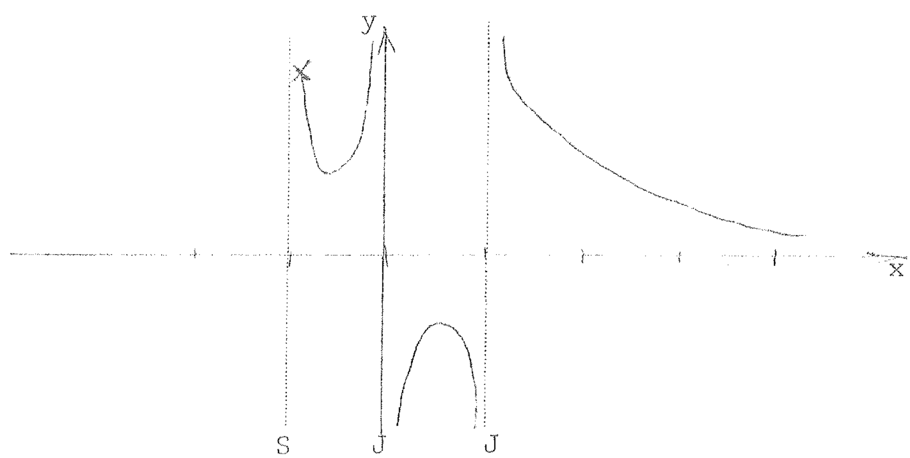
Now the graph comes up, turns around and then approaches the bottom part of the y-axis.



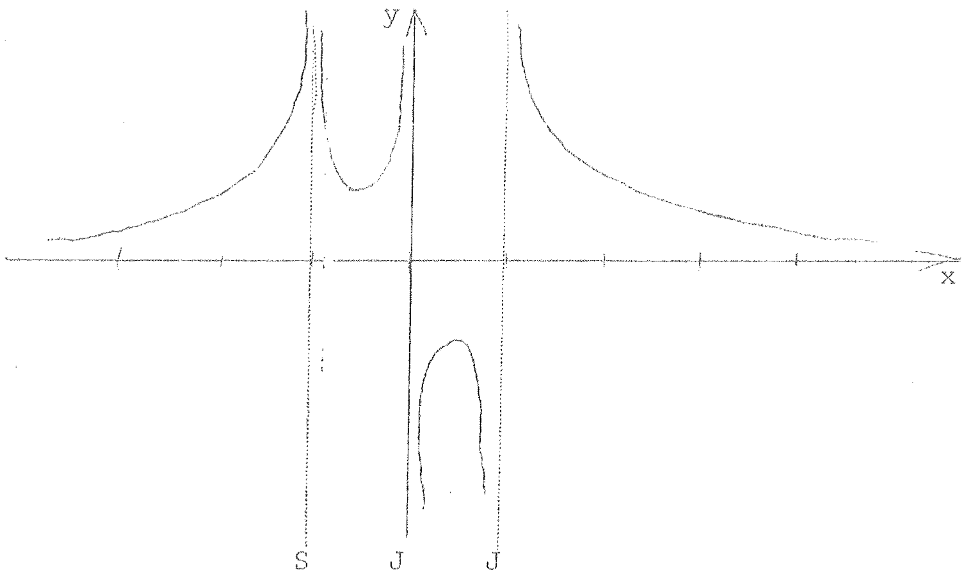
Now, as it is to look like the graph of $y = \pm \frac{1}{x}$ near 0, it jumps to the upper plane near the left side of the y-axis, comes down, turns around and heads up toward the line $x = -1$.



If you wish to check this take an x which is just a tiny bit bigger than -1. Then x is negative, $(x - 1)$ is negative, $x + 1$ is positive and very close to zero. Thus $x(x - 1)^3(x + 1)^2$ is $- \cdot - \cdot +$, and very close to zero, i.e., we are near the cross on the graph below:

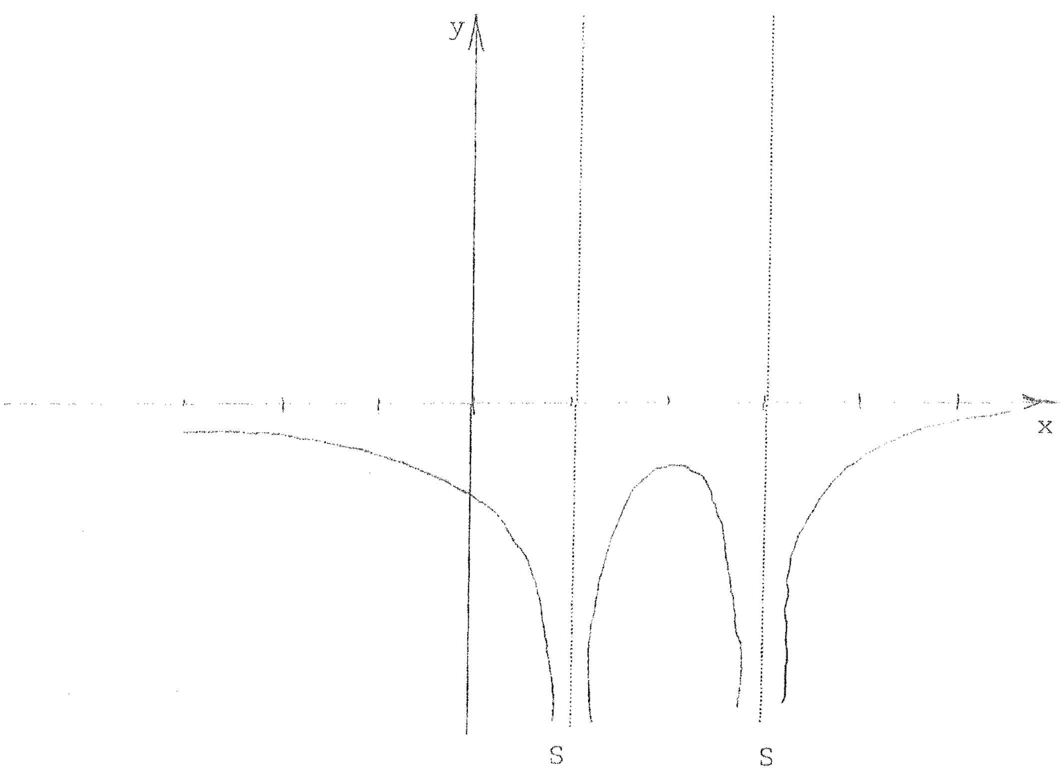


This confirms our previous work. Now the graph stays on the positive end of the asymptote $x = -1$, and then comes down to the x-axis. This is because it must look like $y = \pm \frac{1}{x^2}$ near -1.

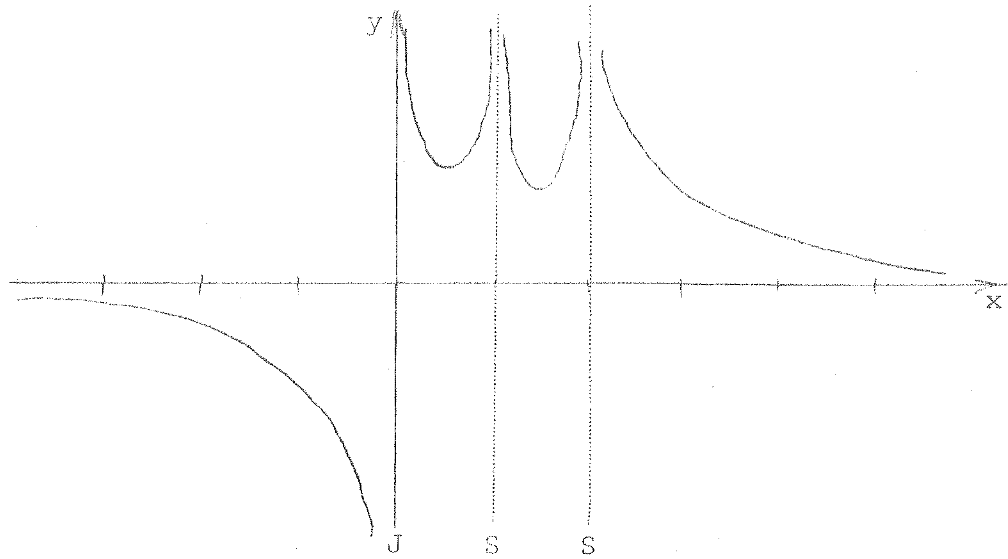


Now we shall do several more examples without any details. You should check them carefully.

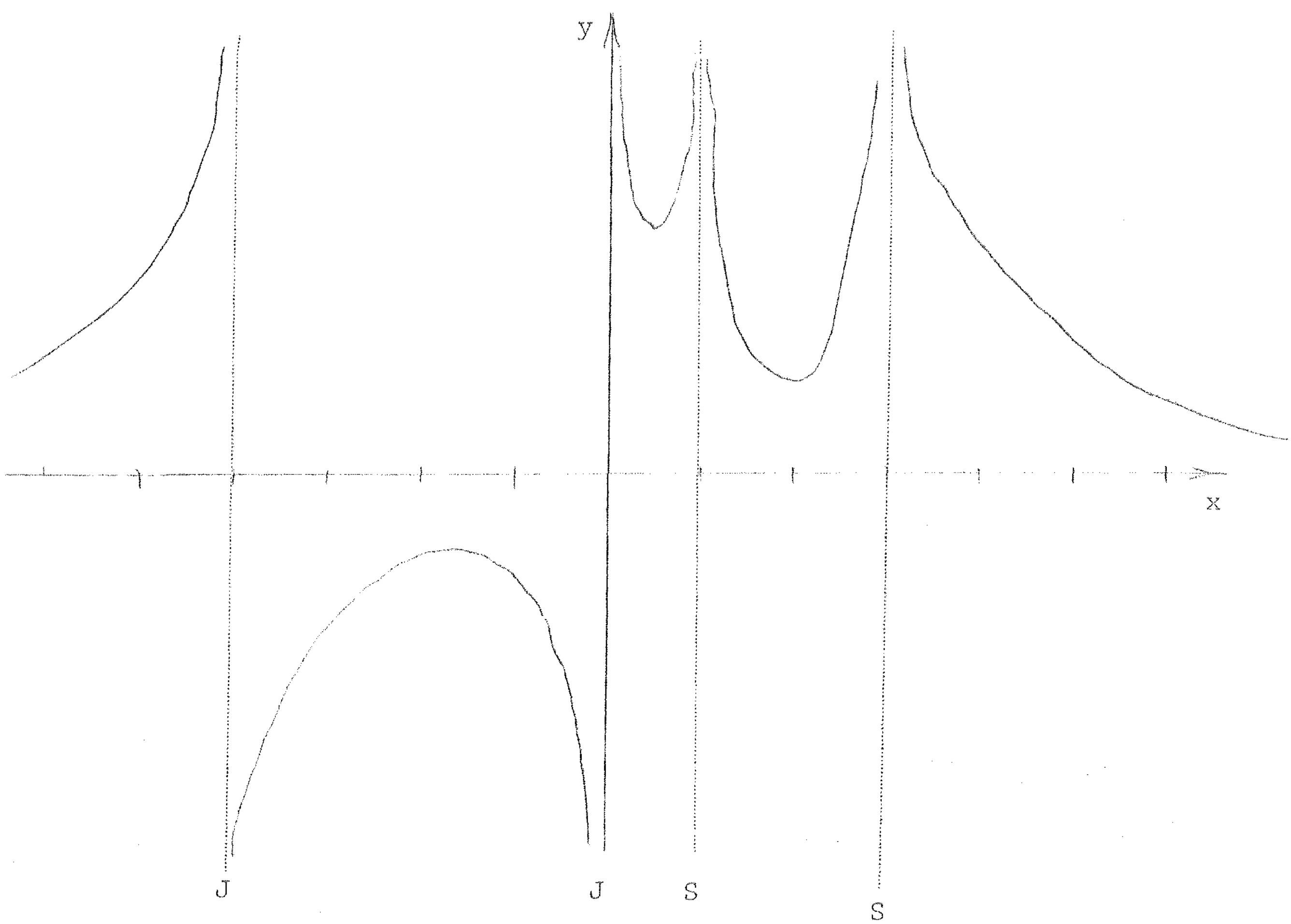
Example 26. $y = \frac{1}{-(x - 1)^2(x - 3)^2}$



Example 27. $y = \dfrac{1}{x^3(x - 1)^2(x - 2)^2}$



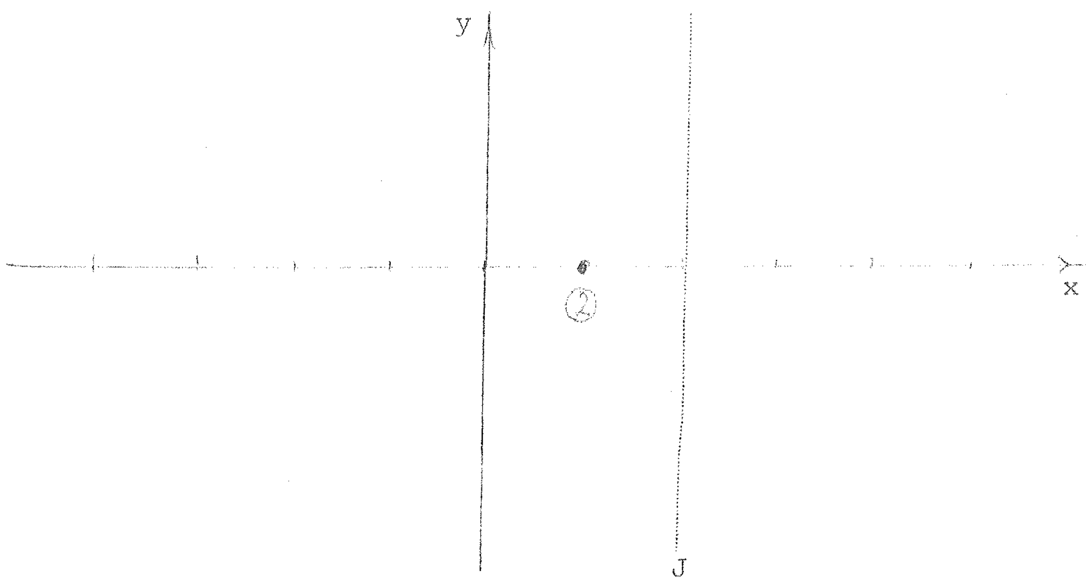
Example 28. $y = \dfrac{1}{x^3(x - 1)^2(x + 4)^3(x - 3)^2}$



In the above examples the numerator has always been 1. The graphs have never intersected the x-axis (although they have "jumped" across near some of the asymptotes). Now we shall introduce the case where the numerator is a polynomial. There are two things to keep in minds. The denominator gives rise to vertical asymptotes and the numerator gives rise to zeroes. These two pieces are combined using the methods previously developed.

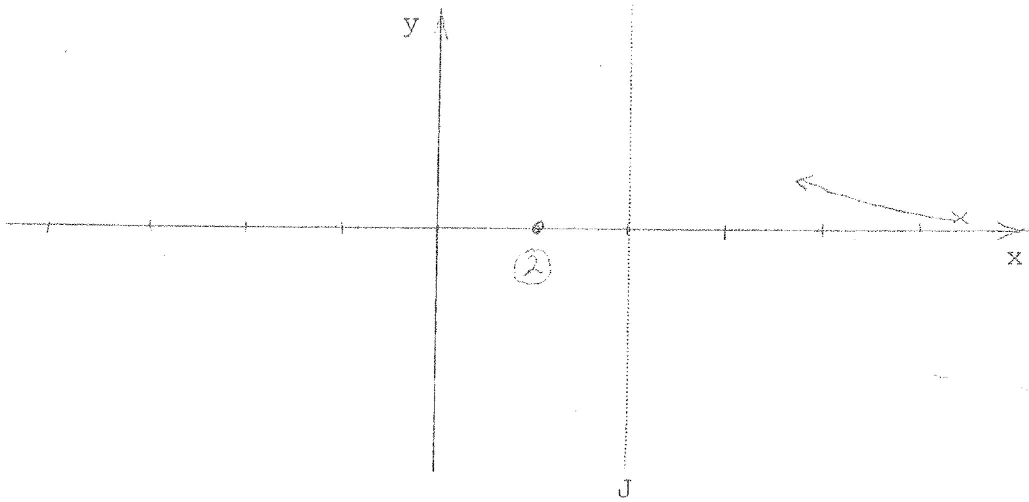
Example 29. $f(x) = \dfrac{(x - 1)^2}{(x - 2)^3}$

This has a vertical asymptote $x = 2$ of the jump type and a zero at 1 of multiplicity 2. This means that near 2 the graph looks like that of $y = \pm \dfrac{1}{x^3}$. Near 1 it looks like $y = \pm x^2$. We begin with

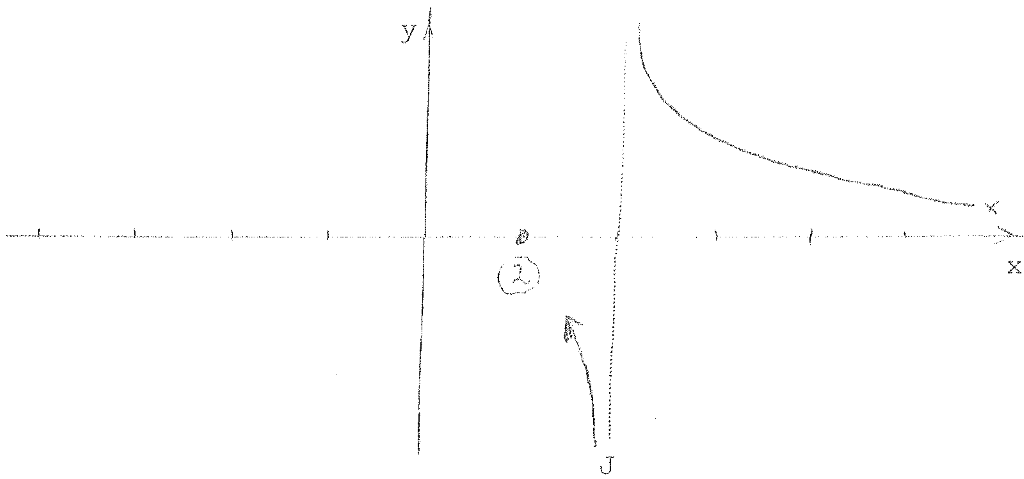


Now we need a starting point. If we put in a very large x then x, $x - 1$ and $x - 2$ are all approximately equal. Thus the numerator is roughly x^2, the denominator x^3.

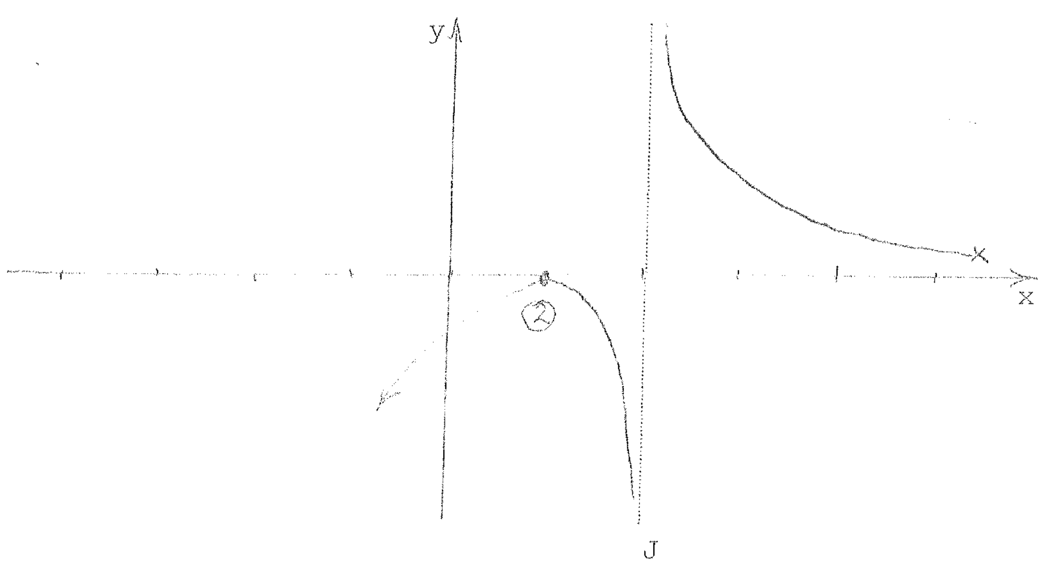
So for large x the value of the function is approximately $\dfrac{x^2}{x^3}$ or $\dfrac{1}{x}$. Thus for large positive x, $f(x)$ is small and positive. The x-axis is a horizontal asymptote and our starting point is marked with a cross:



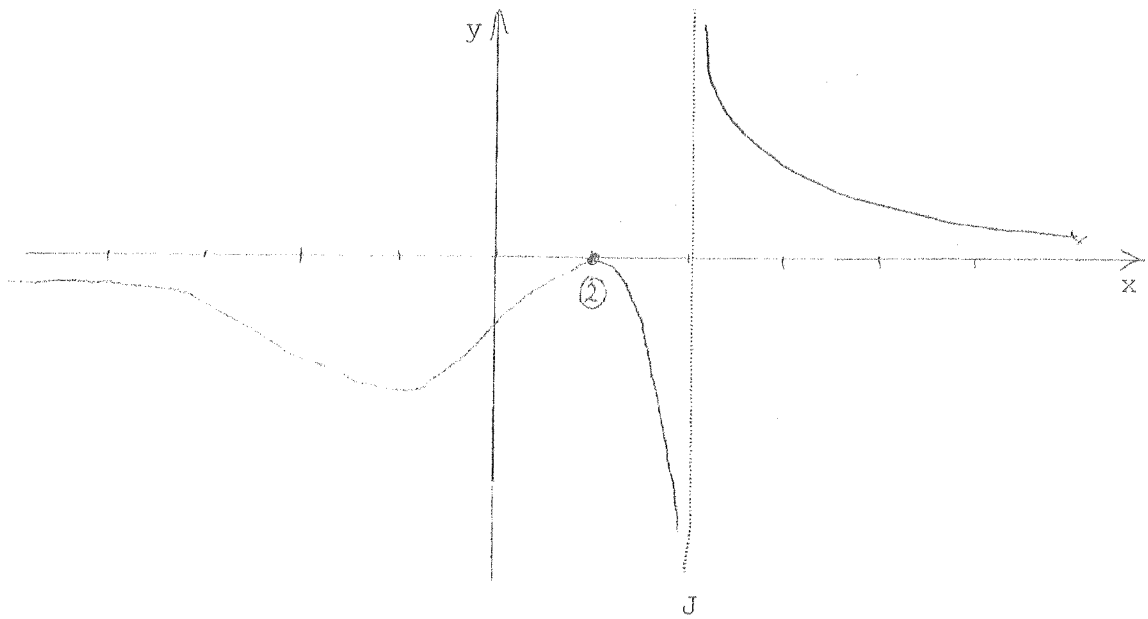
Now near 2 our graph looks like $y = \pm \frac{1}{x^3}$, so, as before, we have:



Now the graph must touch the x-axis and look like the graph of $y = -x^2$:



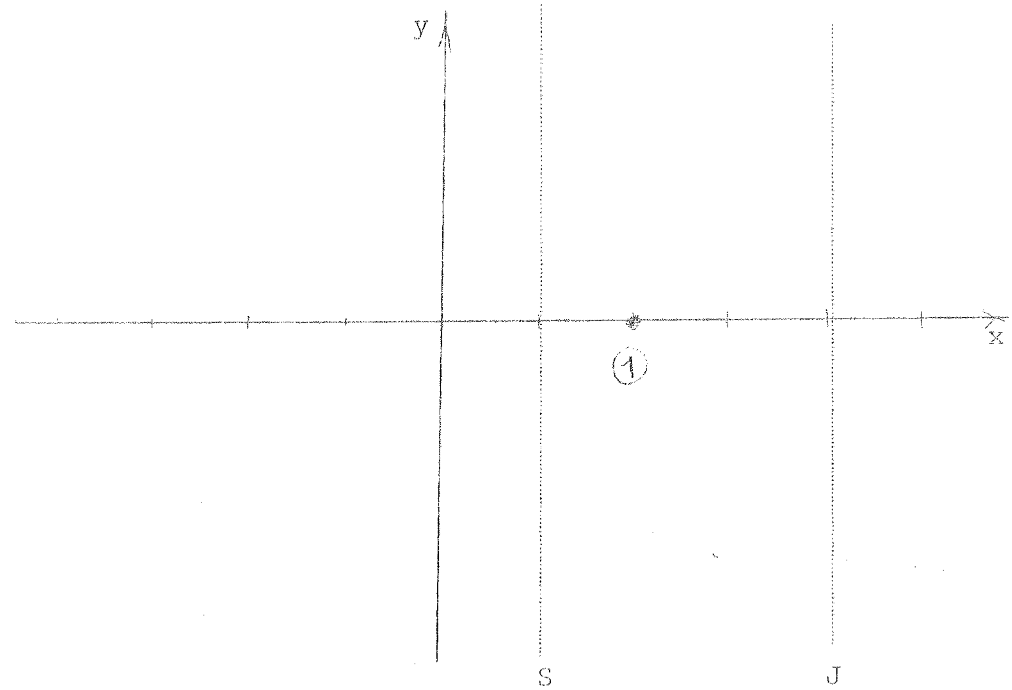
To see where the graph goes next we must examine its behavior when x is large and negative. As before when x is very large $f(x) = \frac{(x-1)^2}{(x-2)^3} \approx \frac{x^2}{x^3} = \frac{1}{x}$. Thus if x is large and negative, $f(x)$ is negative and near zero. The graph is below the negative x-axis, which is a horizontal asymptote. The completed graph is:



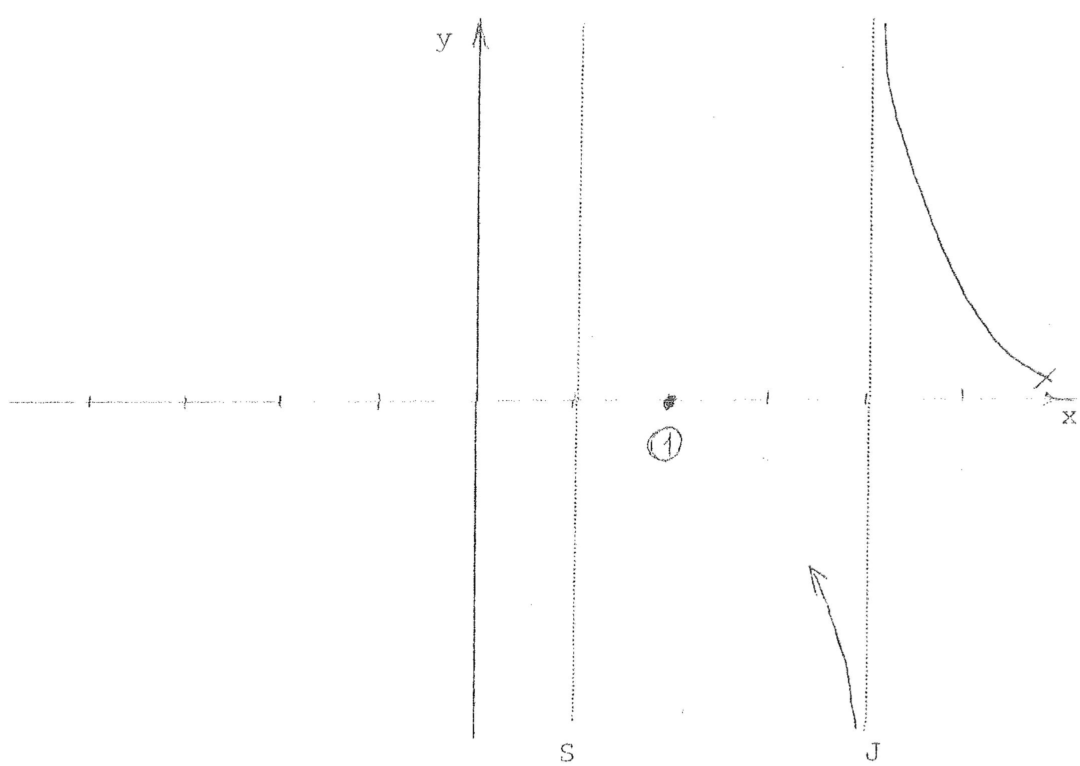
Notice that the curve has a dip in it to the left of the zero at 1. This must be there as the graph touches the x-axis at 2 and it is also asymptotic to the negative x-axis. Later, using calculus, we will locate this minimum point exactly.

Example 30. $f(x) = \frac{(x-2)}{(x-1)^2(x-4)}$

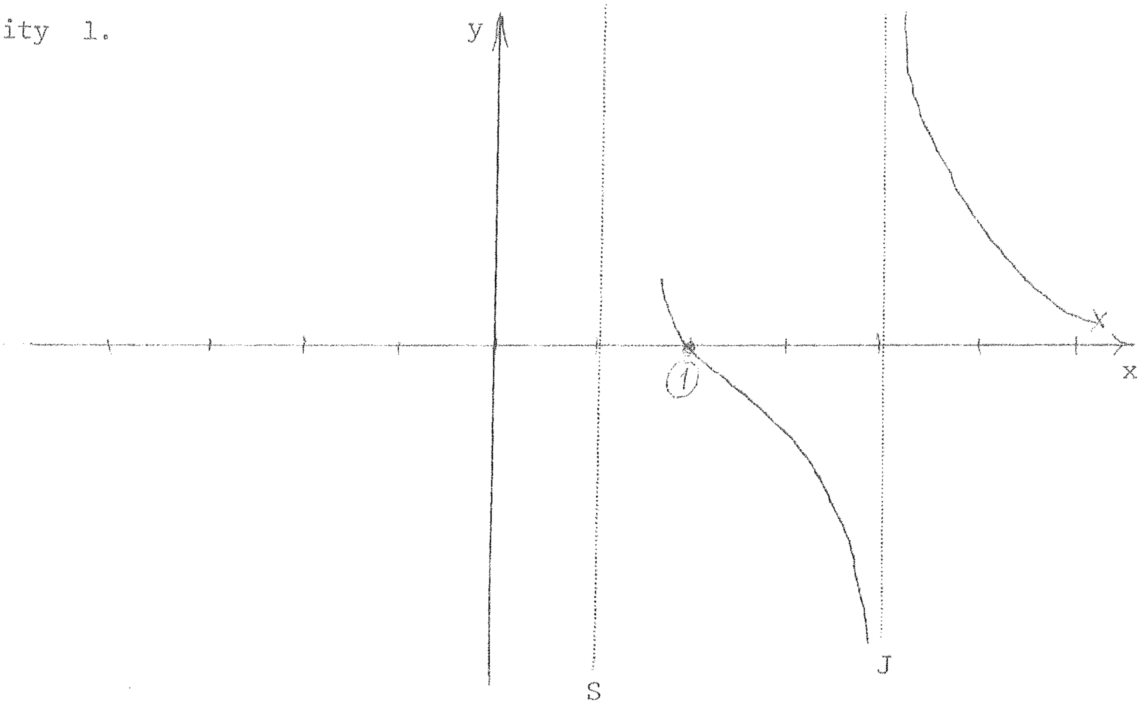
This function has only one zero. That is a zero of multiplicity 1 at 2. So the graph looks like that of $y = \pm x$ near 2. There are two poles. At 1 there is a pole of the S type and at 4 there is one of the J type.



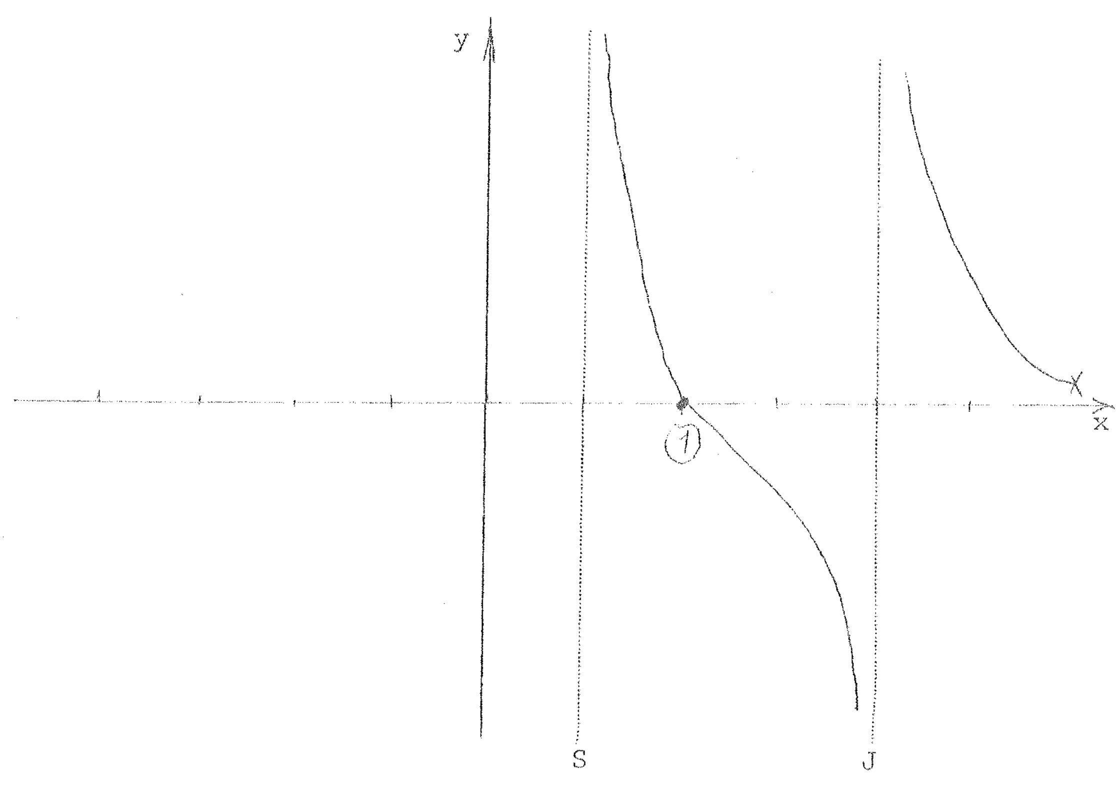
When x is large and positive the numerator is approximately equal to x, while the denominator is roughly x^3. Thus, for large x, $f(x)$ is roughly $\frac{1}{x^2}$, which is very small and positive. This "starting point" is marked and we begin to draw the graph.



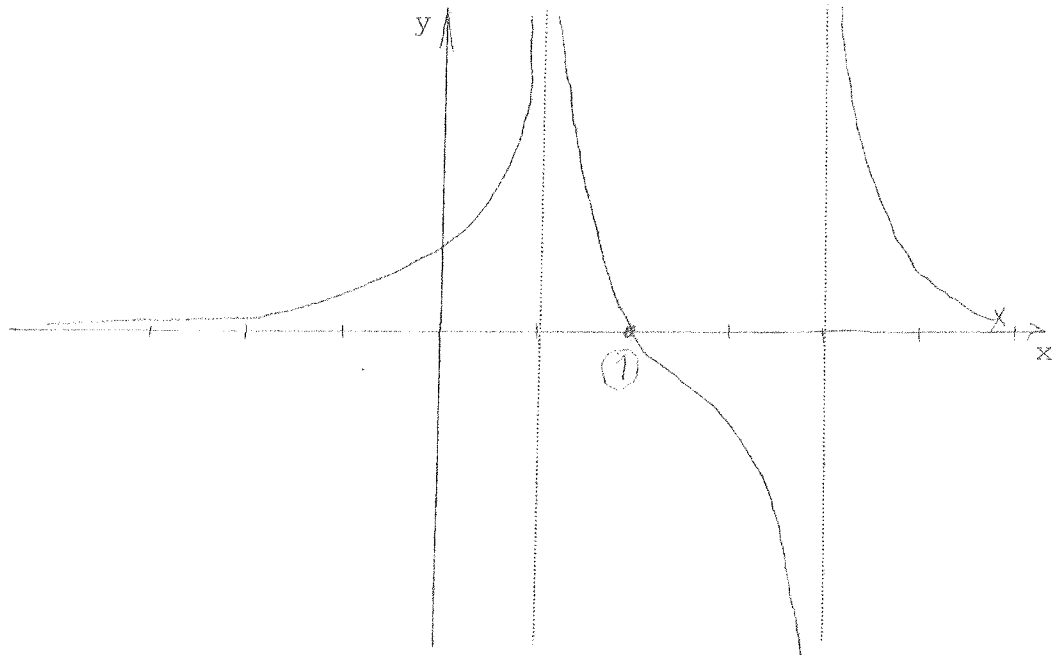
Now the graph must cross the axis at (2,0) as 2 is a zero of multiplicity 1.



Now the graph cannot turn around and cross the axis again as our function has only one zero. So it must head up and approach the positive end of the asymptote x = 1.

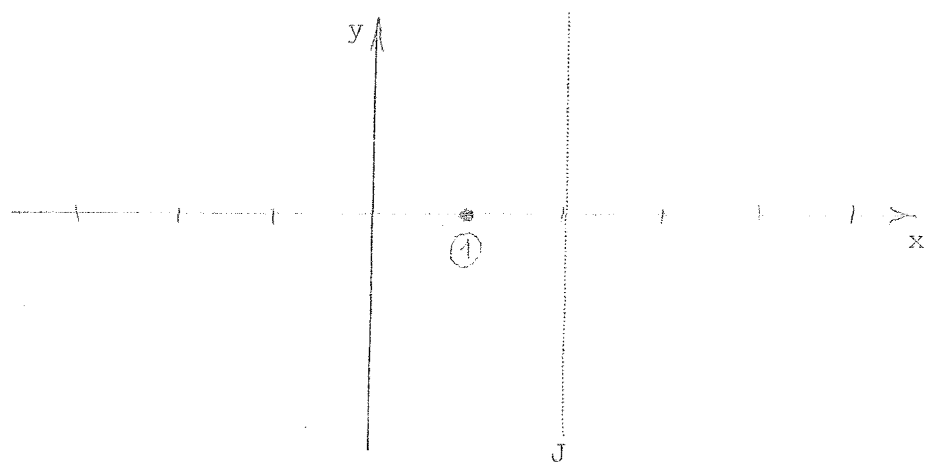


Finally since the asymptote $x = 1$ is of the S type we get our finished graph.



Example 31. $f(x) = \dfrac{x - 1}{x - 2}$

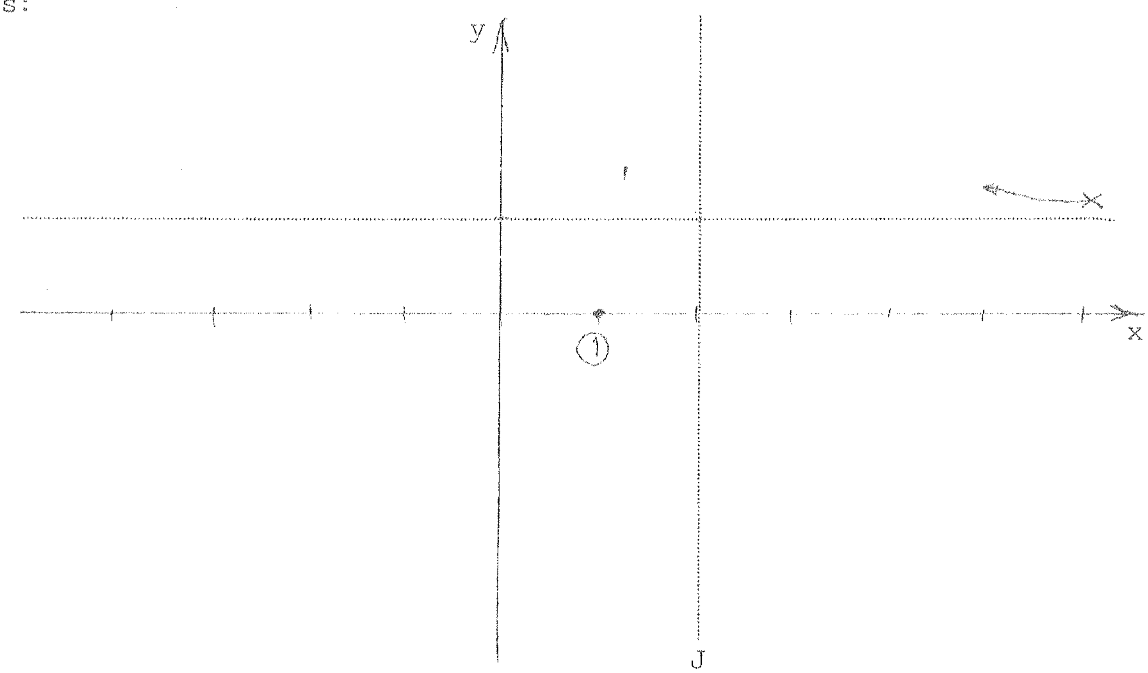
This function has a zero at 1 and a pole at 2:



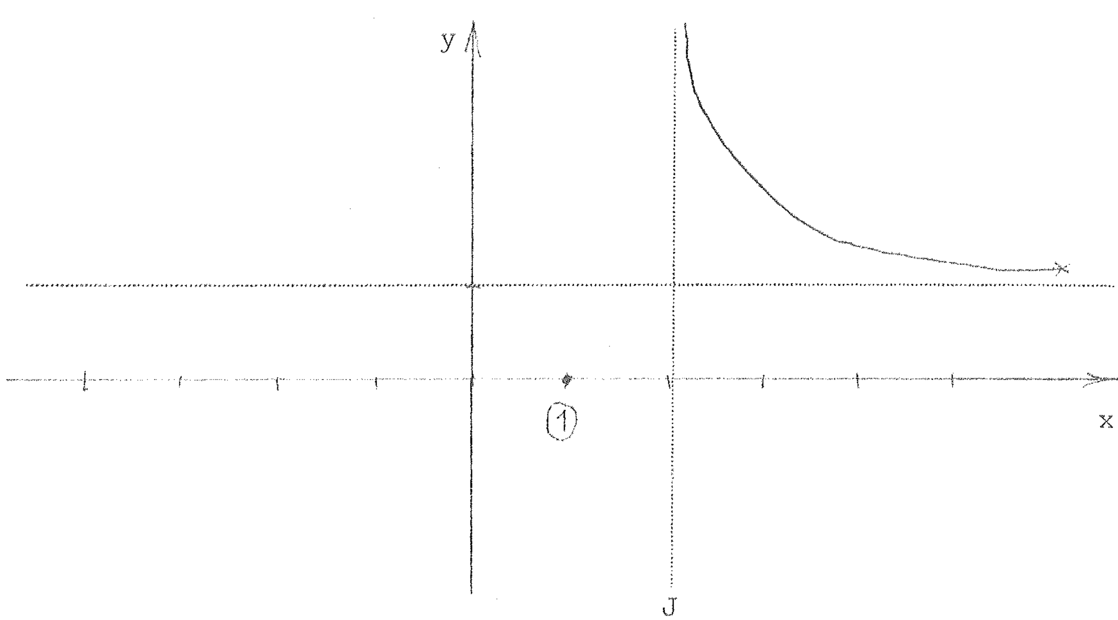
Now we need to find a "starting point." If x is large and positive, then $x - 1$ and $x - 2$ are both positive, so $f(x)$ is positive. Also $x - 1$ and $x - 2$ are almost the same for large x, so $f(x)$ is close to 1. For example, if $x = 1{,}000{,}000$, then $f(1{,}000{,}000) = \dfrac{999{,}999}{999{,}998}$, which is just a little bit bigger than 1. To see this more clearly let us rewrite the function by dividing both numerator and denominator by x:

$$f(x) = \frac{x - 1}{x - 2} = \frac{1 - 1/x}{1 - 2/x} .$$

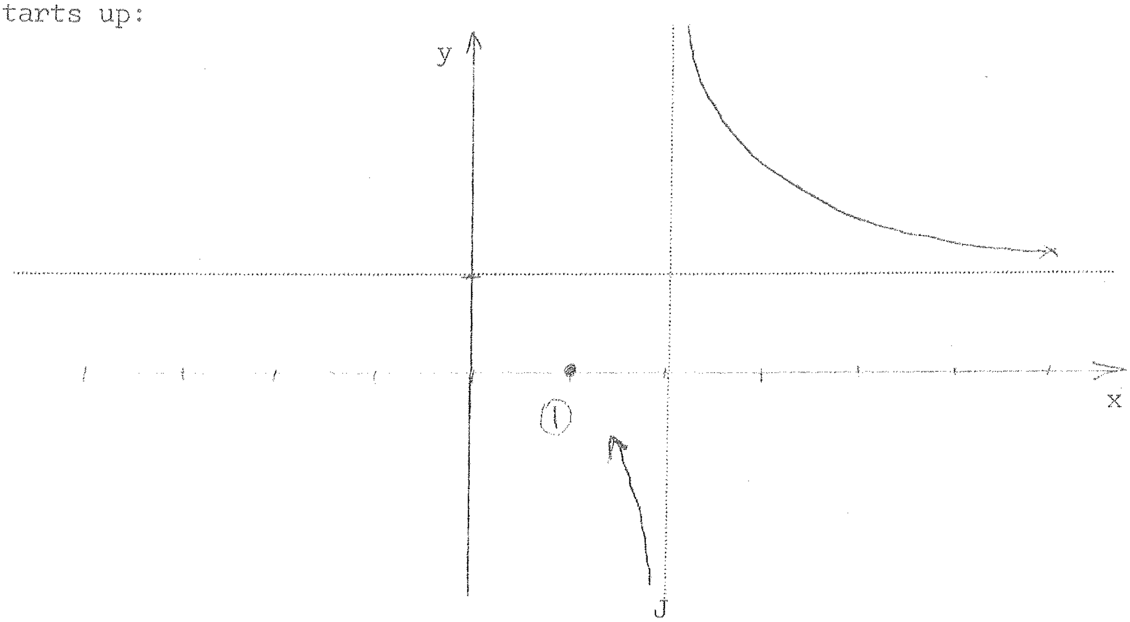
Now as x gets very large, the factors $1/x$ get small and so $f(x)$ gets close to 1. We shall say that the line $y = 1$ is a <u>horizontal asymptote</u>. It is indicated with a dotted line and the starting point is marked with a cross:



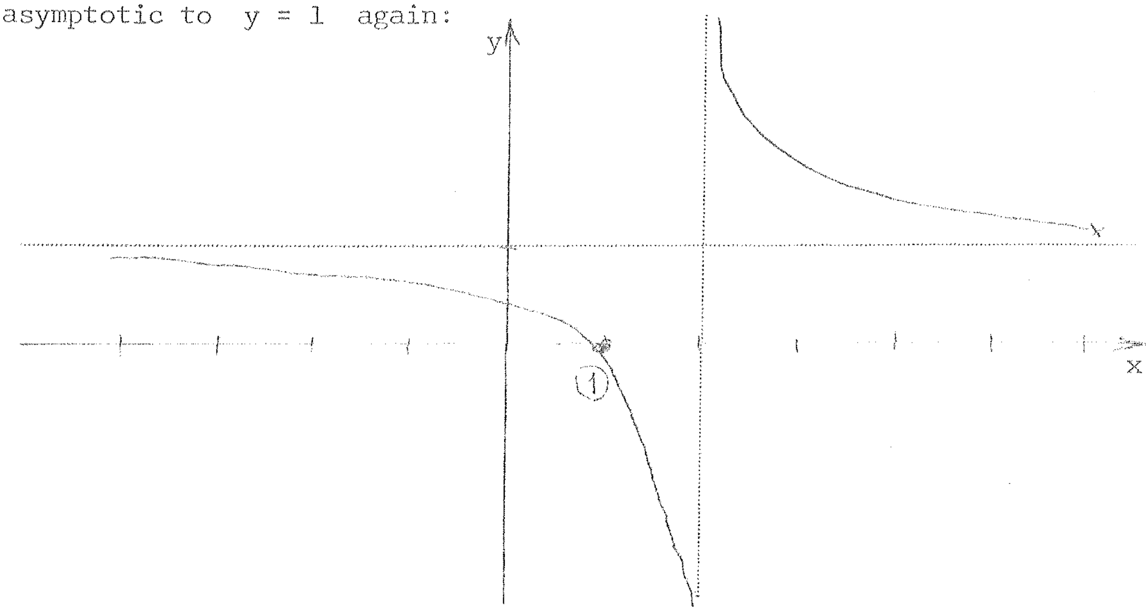
The curve must head up towards the positive end of the asymptote $x = 2$:



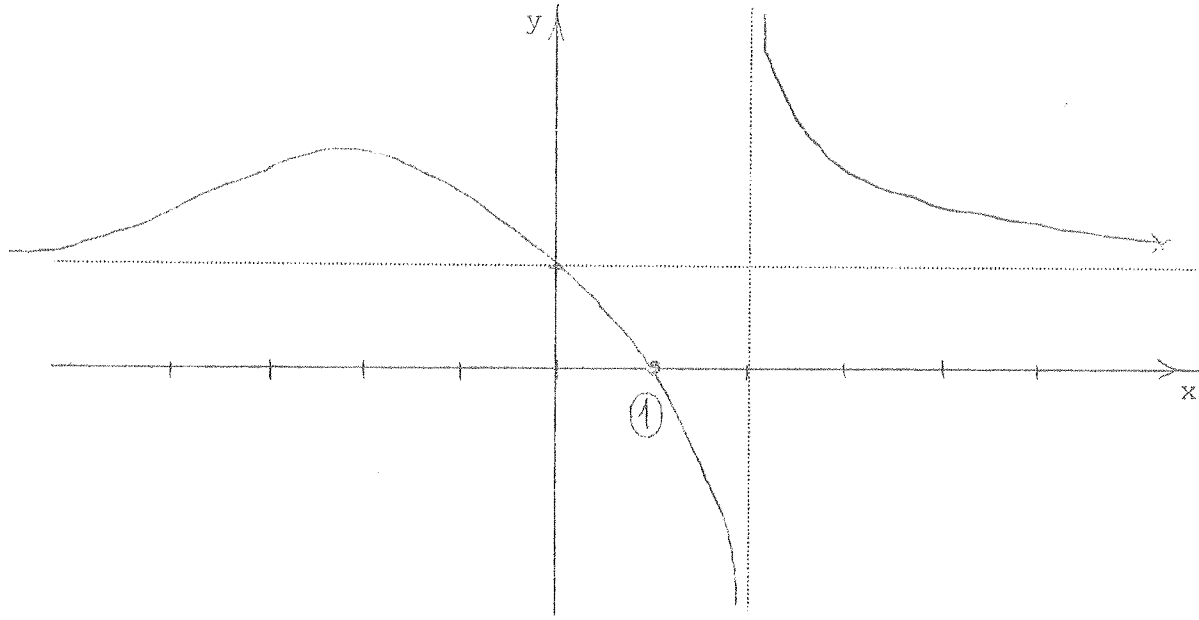
At 2 the curve looks like $y = \pm 1/x$ and so it jumps across the x-axis and starts up:



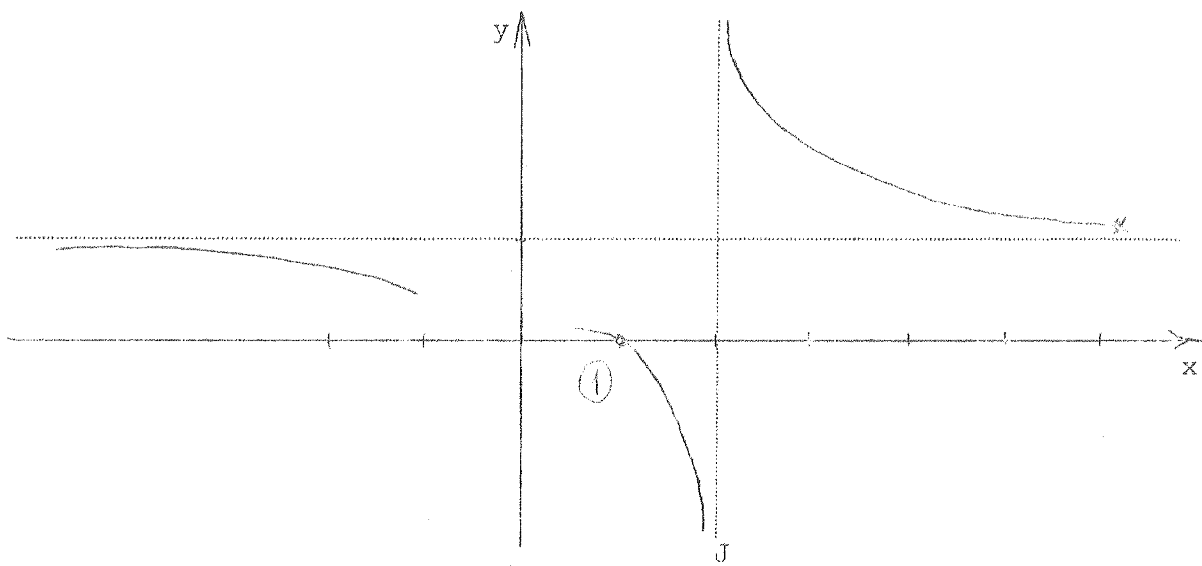
It crosses the axis at (1,0), looking like $y = x$ and then continues to be asymptotic to $y = 1$ again:



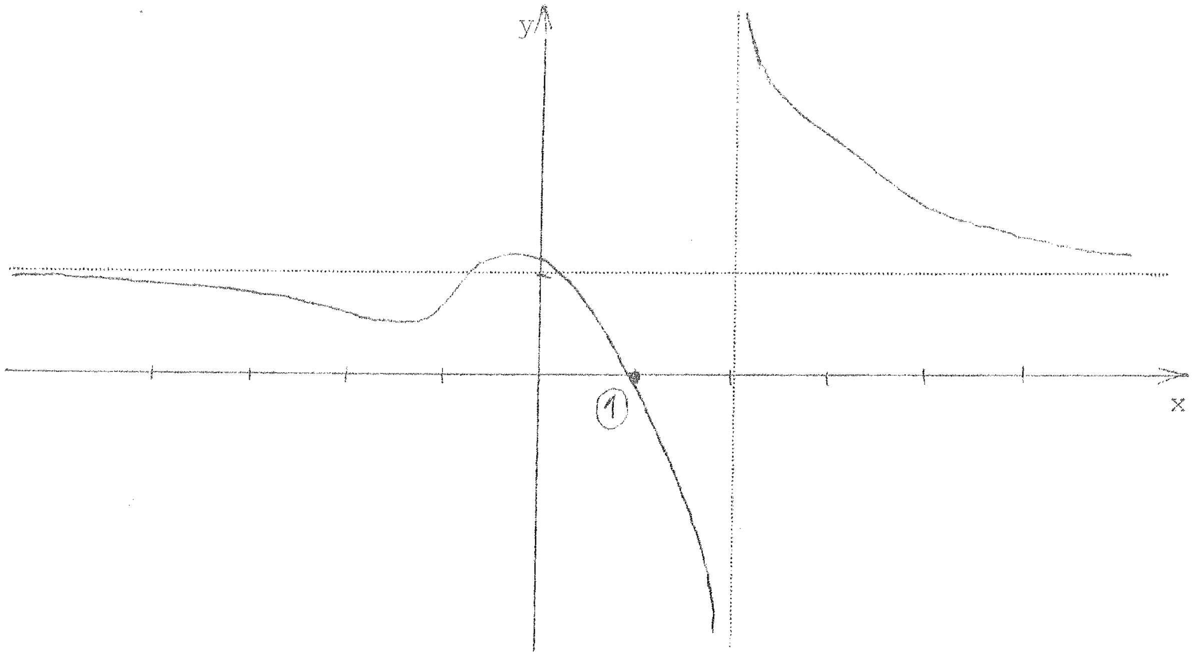
One might wonder why we didn't draw this graph:



The way to explain it is that if x is large and negative then $x - 1 > x - 2$ and so $\dfrac{x-1}{x-2} < 1$. Thus we must have



So if the curve goes above the asymptote after going through $(1,0)$ we must have



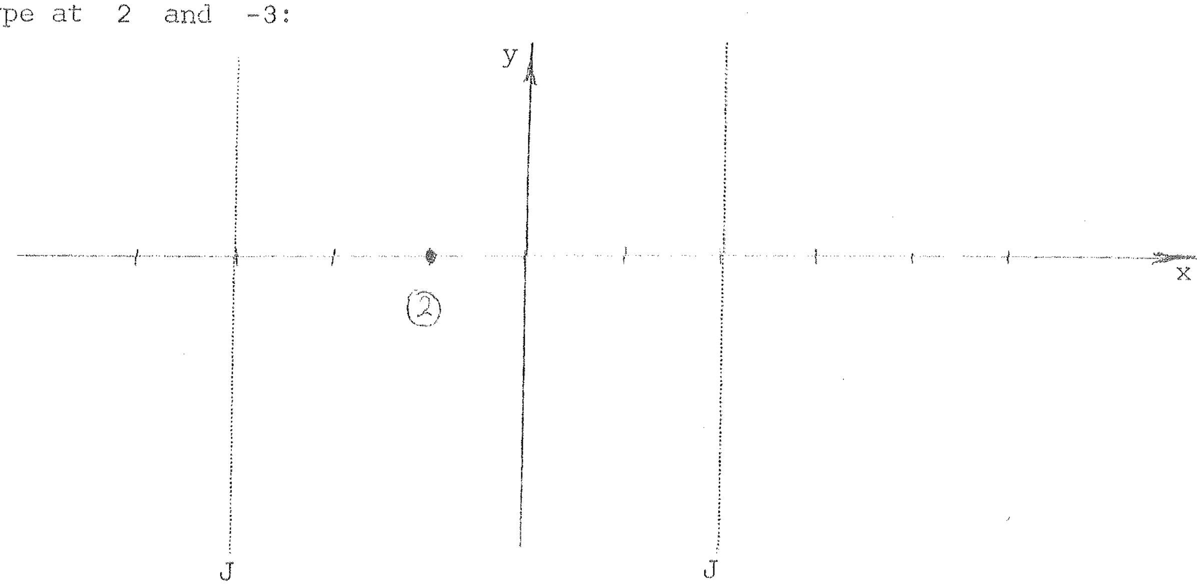
But our curve can't cross the asymptote this many times. The curve intersects the asymptote when $f(x) = 1$, i.e., when

$$\frac{x - 1}{x - 2} = 1 .$$

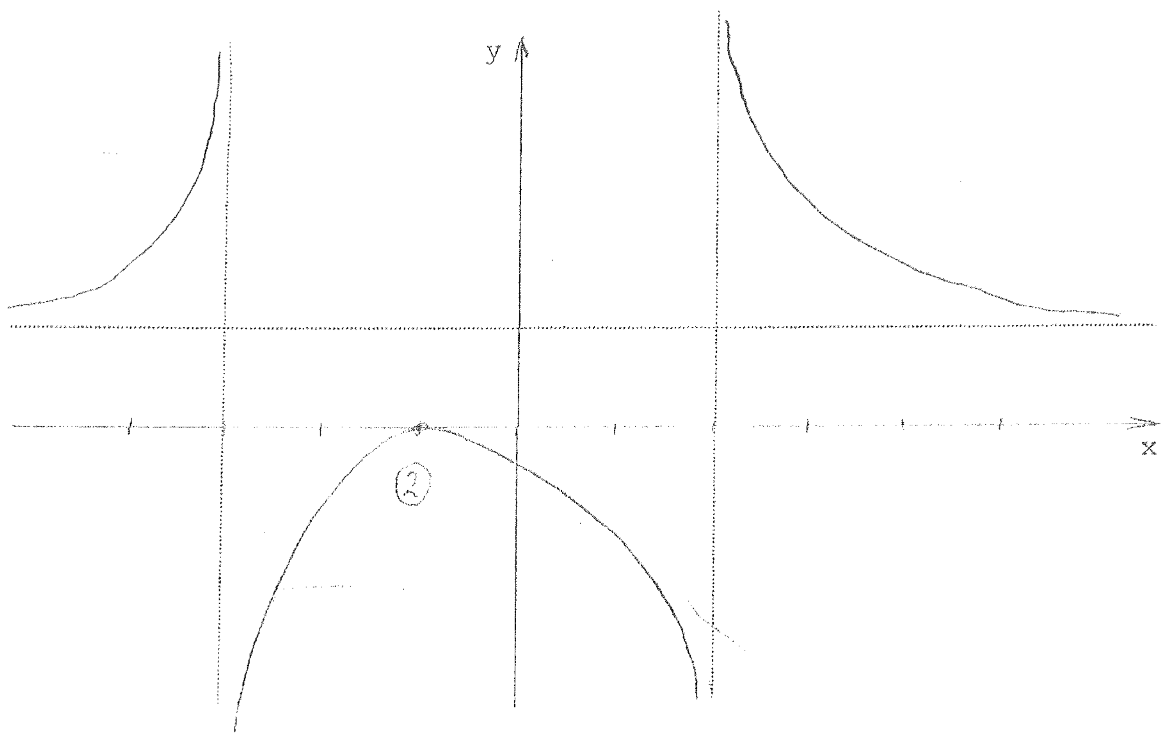
This is true when $x - 1 = x - 2$, i.e., when $-1 = -2$, which never happens. Thus this curve <u>never</u> crosses the horizontal asymptote. Hence the bottom graph on page 44 is the correct one.

<u>Example 32</u>. $f(x) = \dfrac{(x + 1)^2}{(x - 2)(x + 3)}$

This function has a zero of multiplicity 2 at -1 and poles of the jump type at 2 and -3:



To get a starting point note that both the numerator and denominator of $f(x)$ are second degree and thus are very nearly equal for large x. So 1 is a horizontal asymptote. Then when the curve is sketched in using all this information we get:

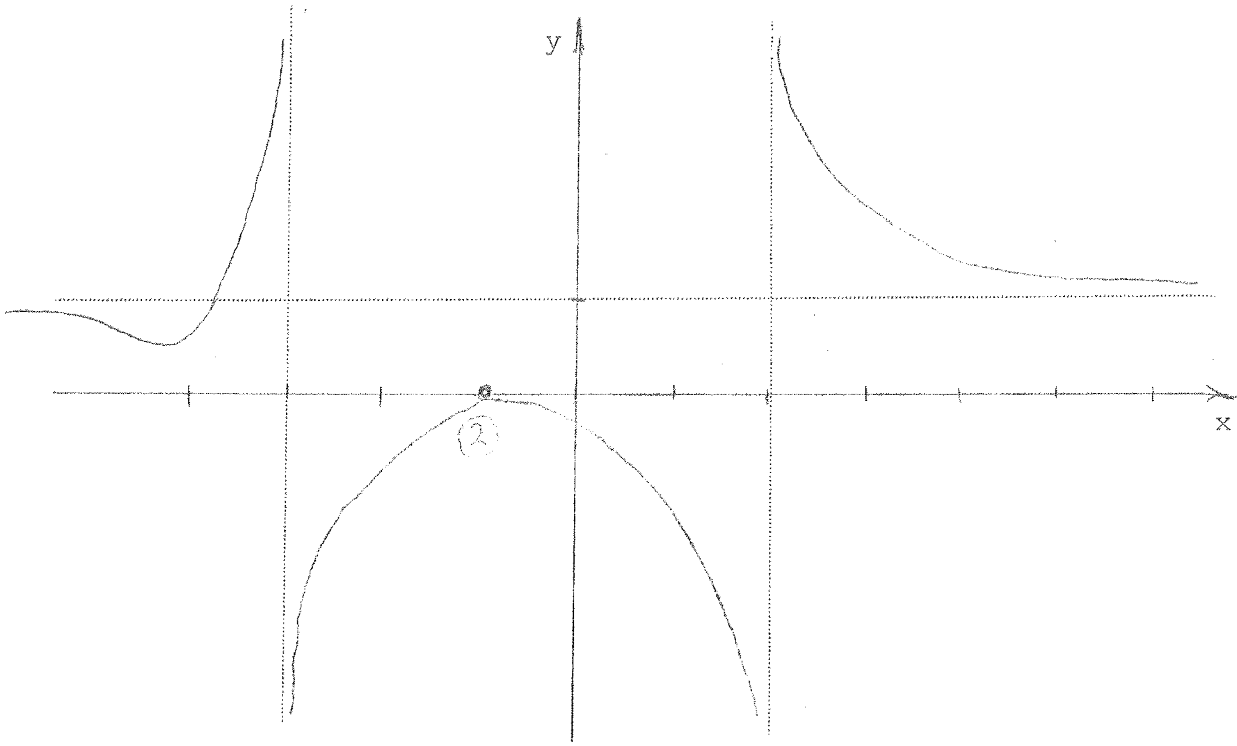


$$f(x) = \frac{(x + 1)^2}{(x - 2)(x + 3)} = \frac{x^2 + 2x + 1}{x^2 + x - 6} = \frac{x + 2 + 1/x}{x + 1 - 6/x}$$

For large positive x this is roughly $\frac{x + 2}{x + 1}$, which is greater than 1 and so we are <u>above</u> the positive end of the horizontal asymptote. This is just as we have drawn it. But if x is large and negative, then

$$\frac{x + 2}{x + 1} < 1 ,$$

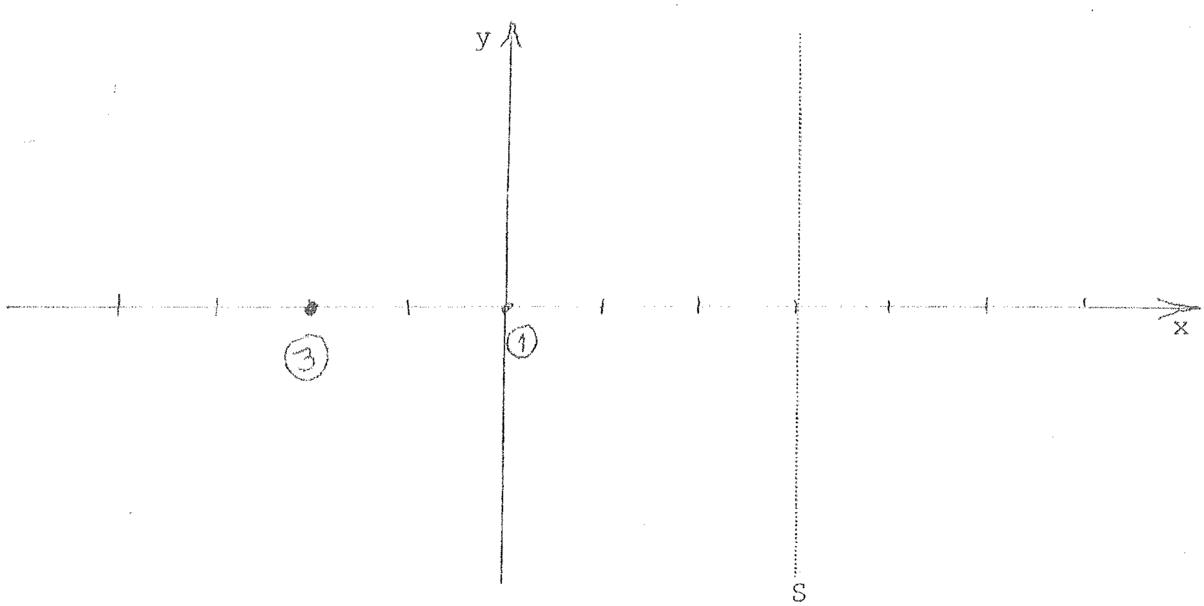
and so the curve should be below the horizontal asymptote $y = 1$. Thus a more accurate graph is



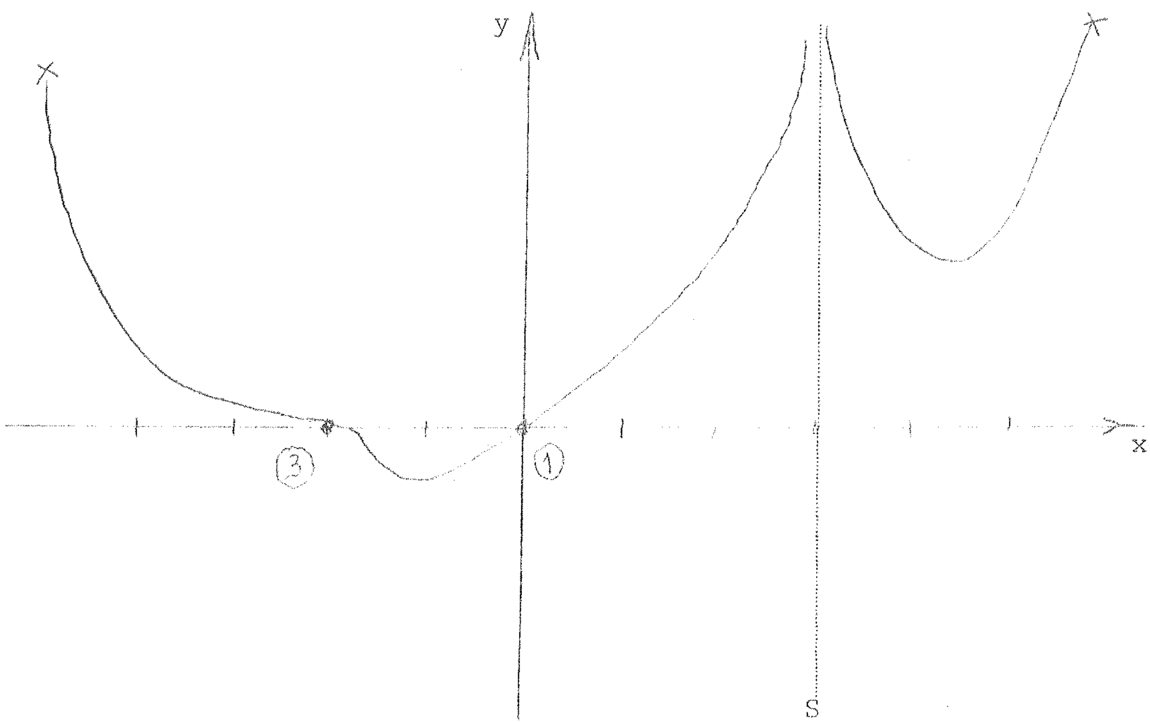
If you wish to find the location of the point where the curve crosses the horizontal asymptote, set $f(x) = 1$. Then $\frac{x^2 + 2x + 1}{x^2 + x - 6} = 1$, i.e., $x^2 + 2x + 1 = x^2 + x - 6$. Thus $2x + 1 = x - 6$, and hence $x = -7$. So the point is $(-7,1)$.

<u>Example</u> <u>33</u>. $f(x) = \frac{x(x + 2)^3}{(x - 3)^2}$

First we record the information about the poles and zeroes on our graph:

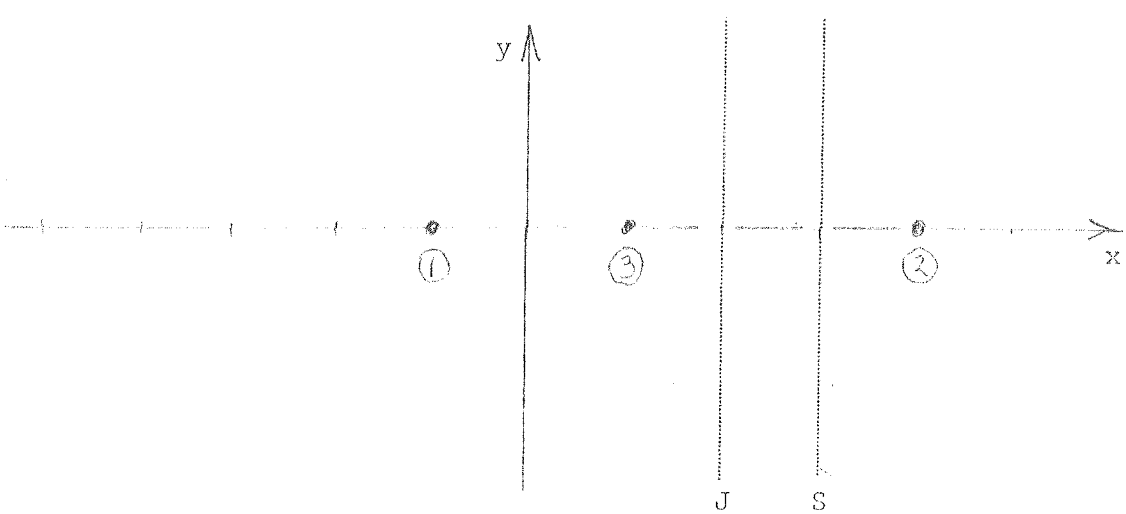


To get started note that if x is very large then the numerator is approximately x^4, while the denominator is approximately x^2. Thus the function behaves like $\frac{x^4}{x^2} = x^2$ for large x. Thus if x is large (either positive or negative), then $f(x)$ is large and positive. We mark both "ends" of our graph with crosses and then draw in the rest of the graph:

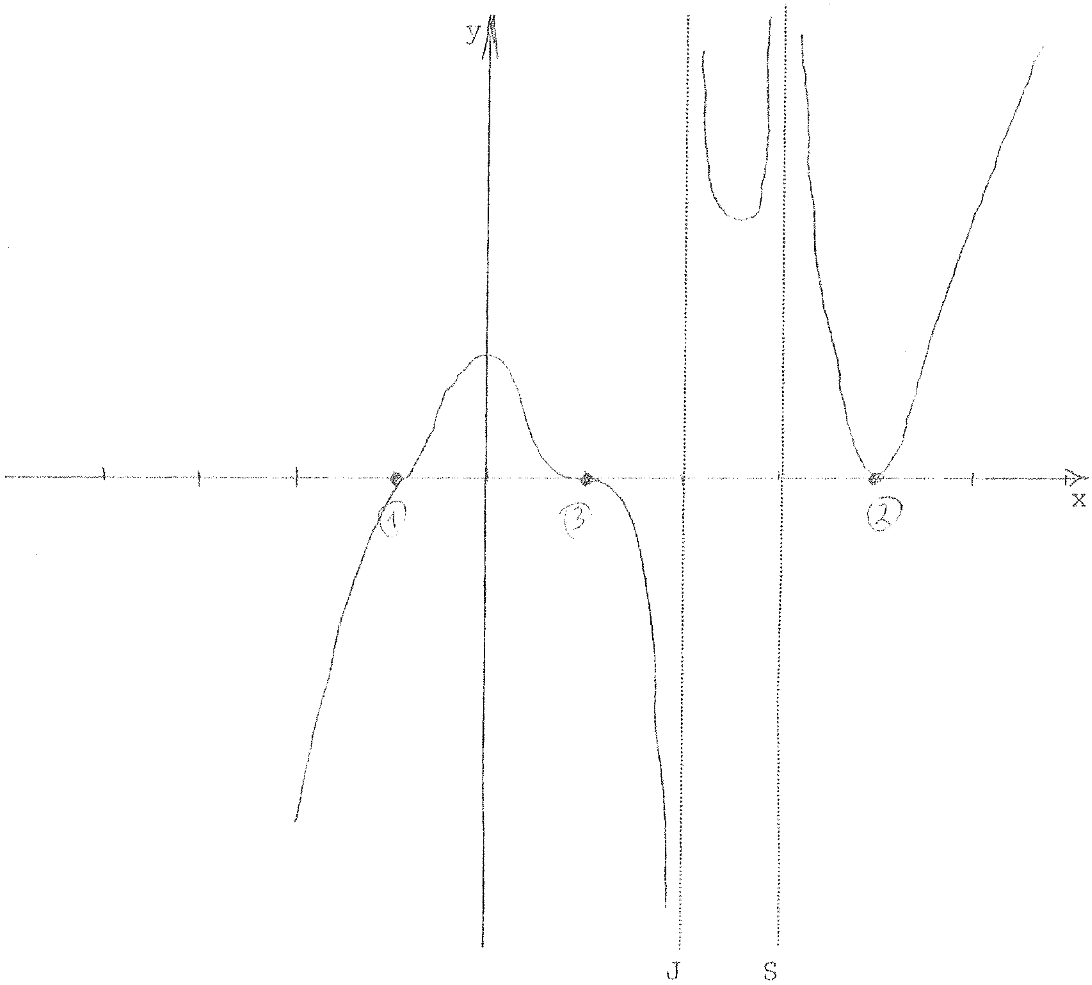


Example 34. $f(x) = \frac{(x - 1)^3(x - 4)^2(x + 1)}{(x - 2)(x - 3)^2}$

Plotting information about zeroes and poles we obtain:



For large x our function is approximately $\frac{x^6}{x^3}$ or x^3. (The thing to remember is that for large x, $x \approx x - 1 \approx x - 4$, etc.) Thus for x large and positive our function is large and positive (and for x large and negative, our function is also). Thus the graph is



If you examine the last six examples carefully you should see a pattern emerging. For some the x-axis is a horizontal asymptote, for others some other horizontal line is a horizontal asymptote, and for still others the function is large for large values of x. Let us look at the problem in general in order to formulate a rule for distinguishing these cases.

The degree of a polynomial is the exponent of the largest power of x which occurs in it, e.g., $-13x^2 + 6x^5$ is of degree 5. We shall see that the degrees of the numerator and denominator determine the behavior of the function in the large, i.e., for large values of x. Thus if you know the degrees of the numerator and denominator you know what the graph looks like for very large x.

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function where p is of degree n and q of degree m. Then our rule is:

If $m > n$, then the x-axis is a horizontal ~~axis~~. asymptote

If $m = n$, then the line $y = \frac{a}{b}$ is a horizontal ~~axis~~, where a (b) is the coefficient of the highest power of x in p (resp. q).

If $m < n$, then there is no horizontal asymptote. For x large $(\pm)$ $f(x)$ is large $(\pm)$ also.

First we illustrate the three cases with the simplest possible examples. $y = \frac{x}{(x-1)^2}$ has the x-axis as a horizontal asymptote. $y = \frac{3x}{x-1}$ has the line $y = 3$ as a horizontal asymptote. $y = \frac{x^2}{x-1}$ has no horizontal asymptote. Now we shall illustrate and explain the rule with more complicated examples.

Example 35. $f(x) = \frac{x(x-2)}{2(x-3)(x+1)}$

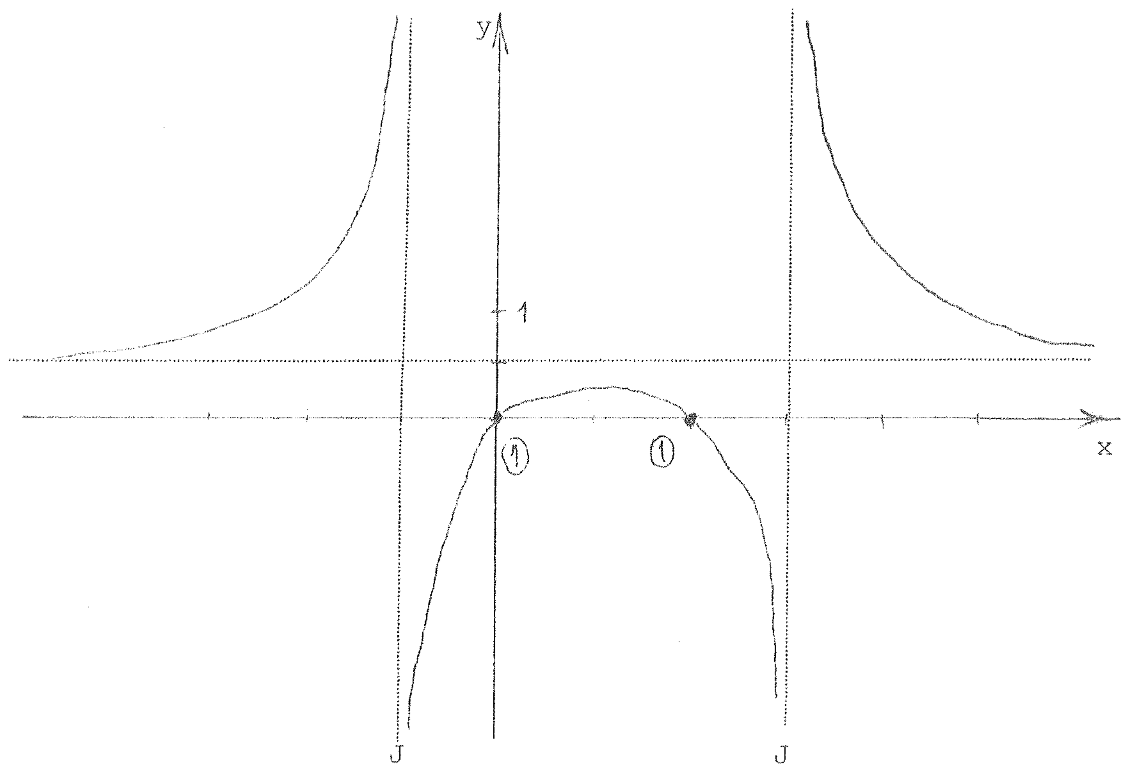
If we multiply out the numerator and denominator we get

$$f(x) = \frac{x^2 - 2x}{2x^2 - 4x - 6}$$

We see that the numerator and denominator have degree 2. (You should be able to see this without actually carrying out the multiplication.) Now if we divide both top and bottom by the highest power of x, i.e., by x^2, we get

$$f(x) = \frac{1 - 2/x}{2 - 4/x - 6/x^2} .$$

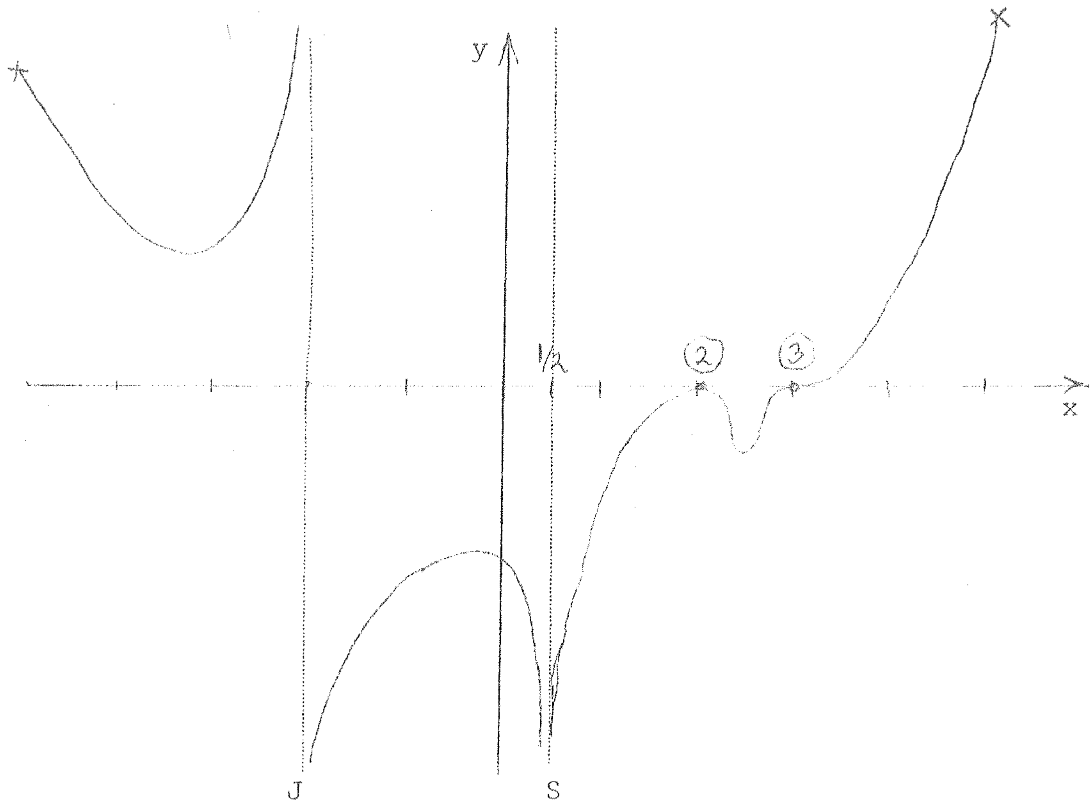
Now if x gets large $2/x$, $4/x$ and $6/x$ all get small, so $f(x)$ gets close to $1/2$. Thus we have a horizontal asymptote at $y = 1/2$. If we indicate this on our graph together with the information about poles and zeroes we have



It is worth remarking that we aren't sure whether the bump between the zeroes goes above the horizontal asymptote or not. (To decide the question solve the equation $f(x) = 1/2$; you will find the curve does not intersect the horizontal asymptote.)

Example 36. $f(x) = \frac{(x - 2)^2(x - 3)^3}{(2x - 1)^2(x + 2)}$

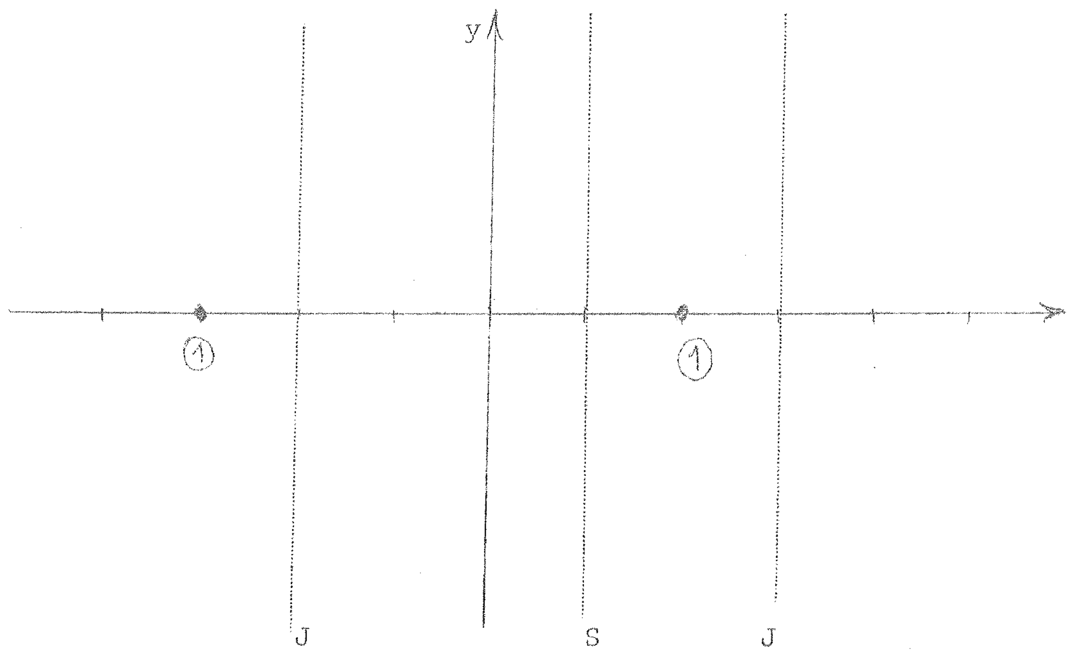
If we multiply out the numerator (denominator) we get a polynomial of degree 5 (3). Thus this function behaves, in the large, like $\frac{x^5}{x^3}$ or x^2, i.e., for large x (either positive or negative), $f(x)$ is large and positive. We obtain the following graph:



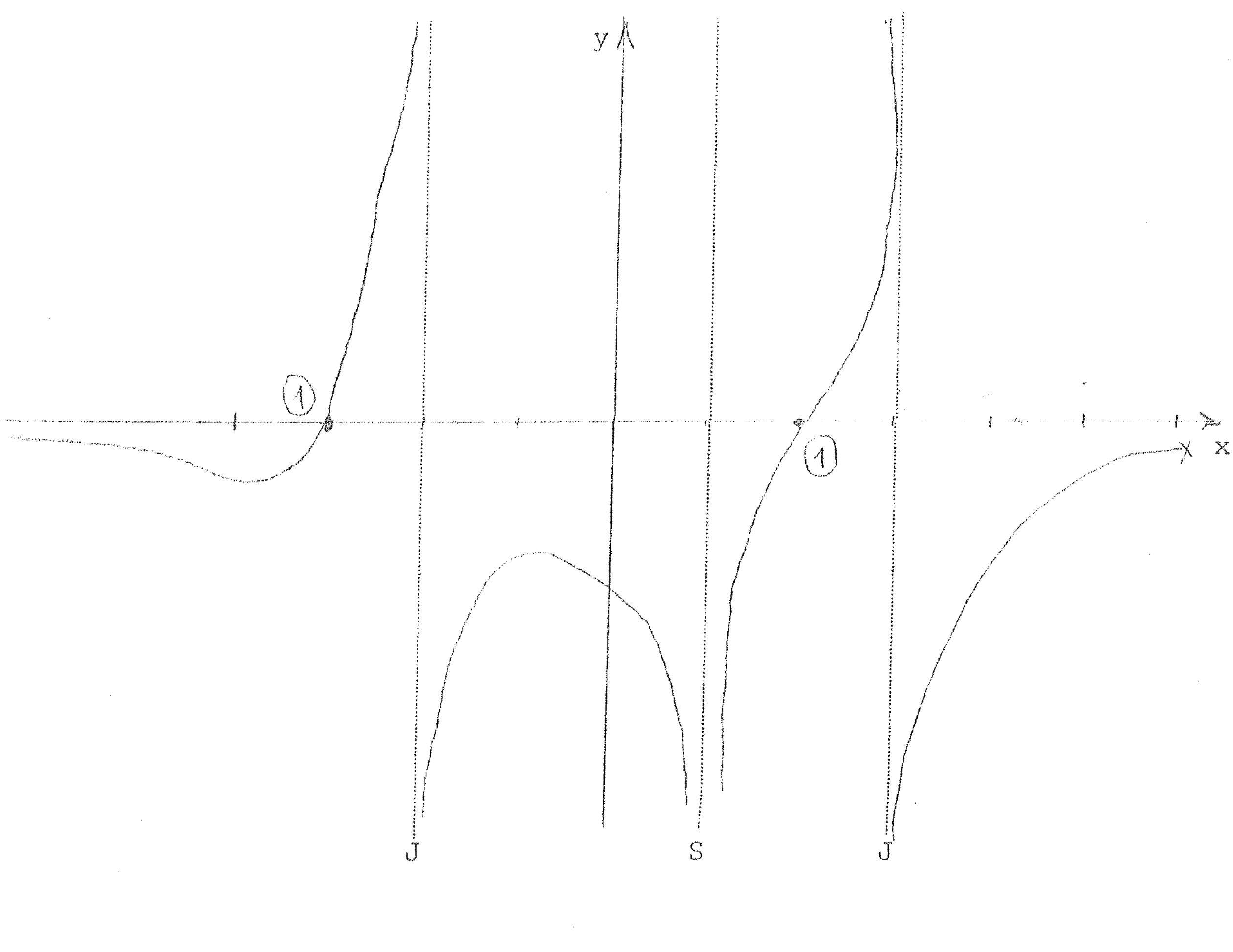
The only thing in need of comment here is the funny factor $(2x - 1)^2$. Remember that this can be written $4(x - 1/2)^2$. Thus there is a vertical asymptote $x = 1/2$ of the stay type.

Example 37. $f(x) = \frac{(x - 2)(x + 3)}{(x - 1)^2(3 - x)^3(x + 2)}$

The numerator is of degree 2 and the denominator is of degree 6, so the x-axis is a horizontal asymptote (the function behaves in the large like $y = \frac{1}{x^4}$). Plotting information about zeroes and poles we have:

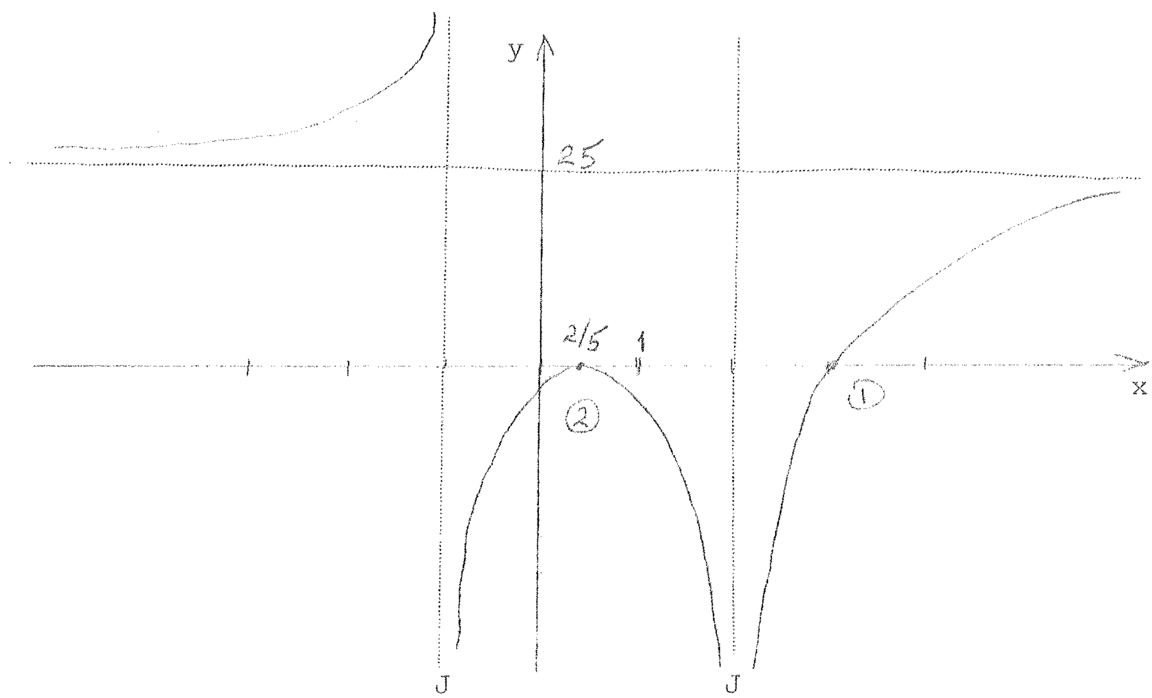


When we go to find a starting point we must be careful. If x is large and positive, then the numerator is also, but the denominator is large and negative (this is due to the factor $(3 - x)^3$). Thus $f(x)$ is small, but negative. So, with the starting point indicated by a cross, the graph is:



Example 38. $f(x) = \dfrac{(5x - 2)^2(x - 3)}{(x - 2)^2(x + 1)}$

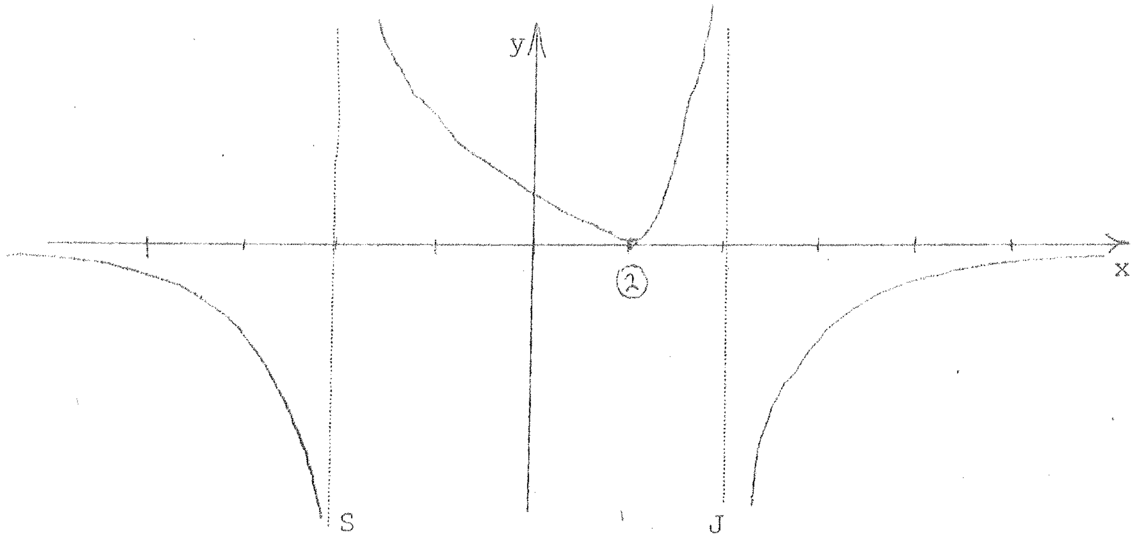
Here the numerator and denominator are both of degree 3, so there is a horizontal asymptote. To find it we must ascertain the coefficients of x^3 in the numerator and denominator. In the numerator it is 25 while it is 1 in the denominator. (In the first few examples you do you may have to multiply things out, but that should not persist.) Thus $y = 25$ is a horizontal asymptote. The graph is thus



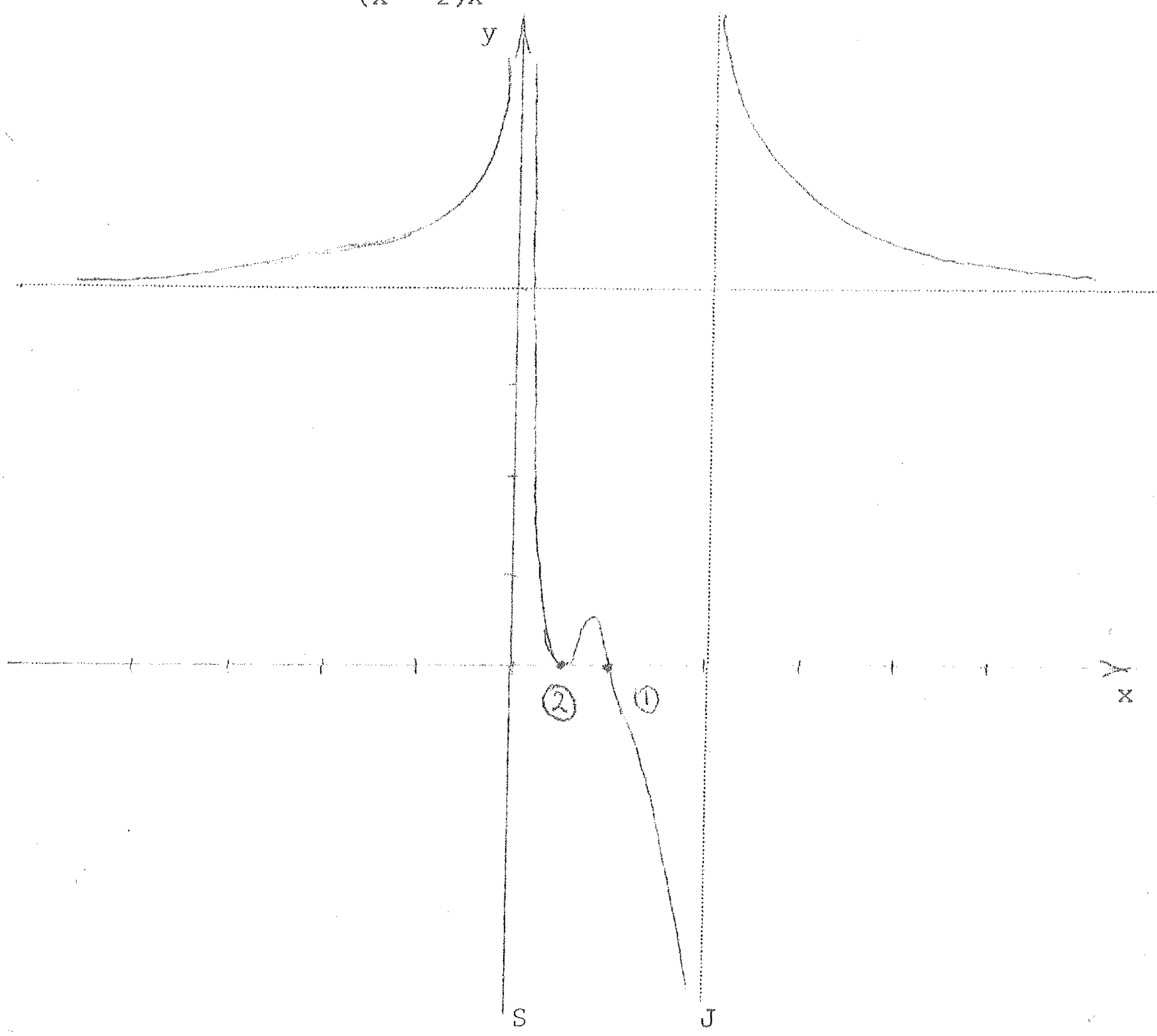
The point needing comment is the zero arising from the factor $(5x - 2)^2$. This is zero when $5x = 2$, i.e., when $x = 2/5$. Note that our graph has the point $(2/5,0)$ as a zero of multiplicity 2.

We shall close with a few more examples without comment. You should check that they are done correctly.

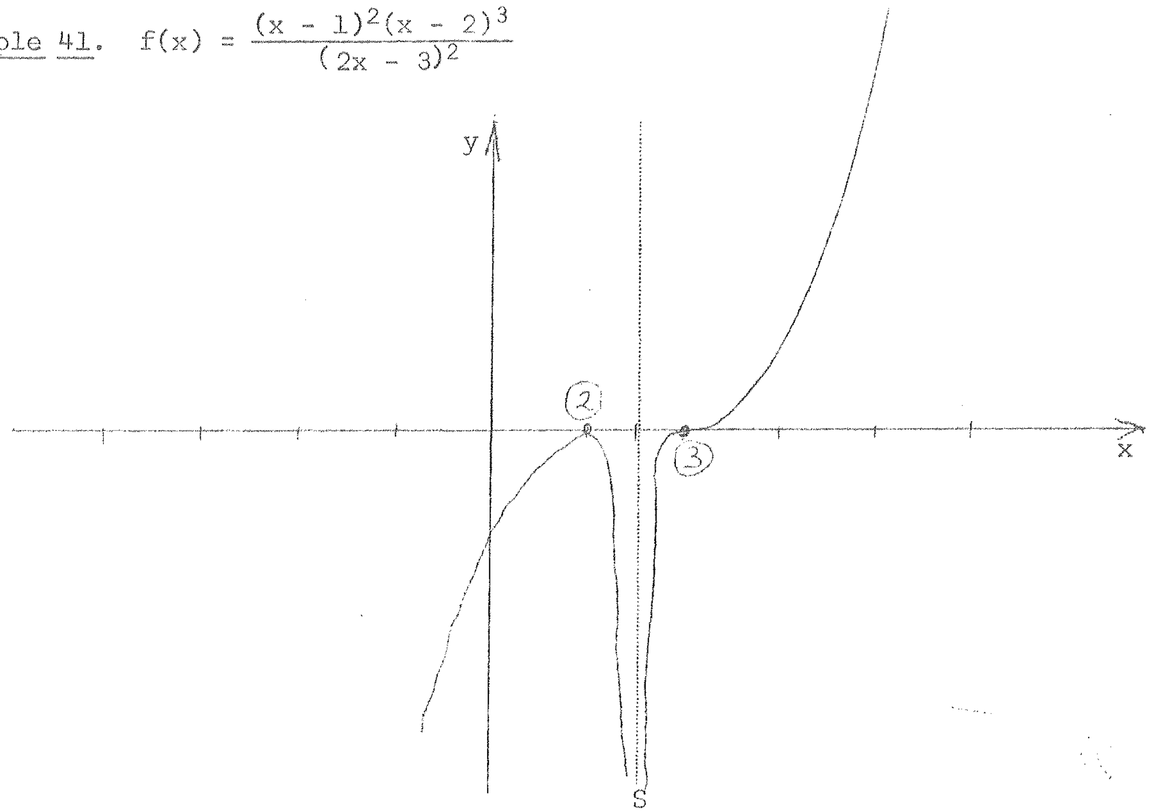
Example 39. $f(x) = \dfrac{(x - 1)^2}{(2 - x)^3(x + 2)^{\cancel{2}3}}$



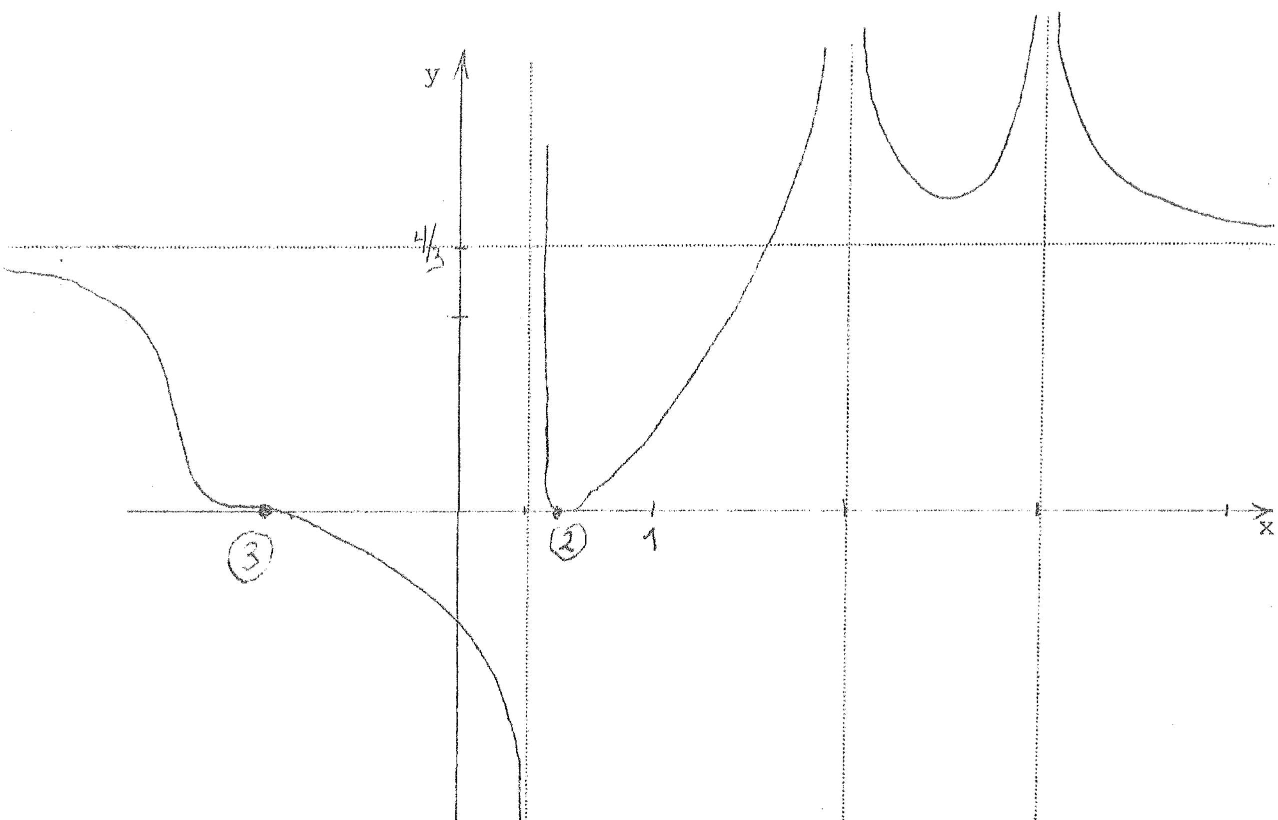
Example 40. $f(x) = \dfrac{(2x - 1)^2(x - 1)}{(x - 2)x^2}$



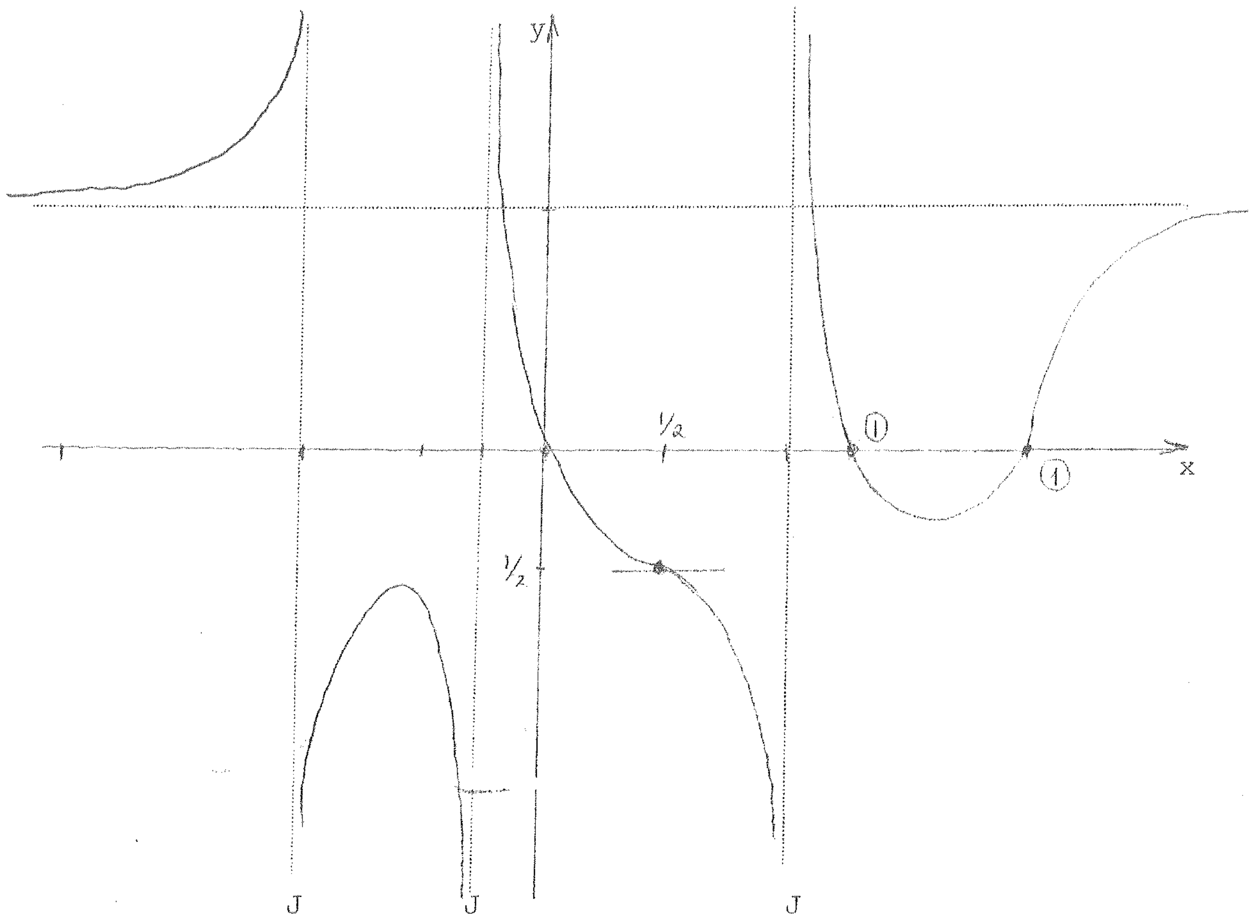
Example 41. $f(x) = \dfrac{(x-1)^2(x-2)^3}{(2x-3)^2}$



Example 42. $f(x) = \dfrac{(2x-1)^2(x+1)^3}{(3x-1)(x-2)^2(x-3)^2}$



Example 43. $f(x) = \dfrac{(x - 5/4)(x - 2)x}{(x - 1)(x + 1)(x + 1/4)}$



Remark. The correct graph has been drawn. It must be mentioned that the horizontal tangent at $(1/2, 1/2)$ cannot be determined by the methods outlined here. You must use the calculus. This example shows that our methods do not capture every important qualitative property of the graphs of rational functions. The calculus is necessary to fine-tune the graphs.

EXERCISES:

Part I

Graph each of the following rational functions. You should pay particular attention to the behavior of the functions near the vertical asymptotes. Near a vertical asymptote the function can either stay on the same side of the x-axis or jump across the x-axis. Indicate on your graphs whether the vertical asymptotes are of the jump (J) or stay (S) type.

In the first group of problems the graphs neither touch nor cross the x-axis, as these functions have no zeroes. Also, in this first group, the x-axis is a horizontal asymptote.

1. $f(x) = \dfrac{1}{x - 3}$

2. $f(x) = \dfrac{1}{(x - 3)^2}$

3. $f(x) = \dfrac{1}{3 - x}$

4. $f(x) = \dfrac{1}{(3 - x)^2}$

5. $f(x) = \dfrac{1}{x(x - 3)}$

6. $f(x) = \dfrac{1}{x^2(x - 3)^3}$

7. $f(x) = \dfrac{1}{(x - 2)(x + 4)}$

8. $f(x) = \dfrac{-1}{(x - 2)(x + 2)}$

9. $f(x) = \dfrac{1}{(2 - x)(x + 4)}$

10. $f(x) = \dfrac{1}{x^3 - x^2}$

11. $f(x) = \dfrac{1}{(x - 3)^2(x - 1)}$

12. $f(x) = \dfrac{1}{(x + 2)^2(x - 3)}$

13. $f(x) = \dfrac{1}{(x - 1)^2(x + 1)^3}$

14. $f(x) = \dfrac{1}{(x + 1)(x - 1)^3}$

15. $f(x) = \dfrac{1}{x(x - 1)(x - 2)}$

16. $f(x) = \dfrac{1}{x^2(x - 1)(x - 2)^3}$

17. $f(x) = \dfrac{1}{x^3(x + 1)(x - 1)^2}$

18. $f(x) = \dfrac{1}{(x - 2)^3(x - 1)^2(x + 3)}$

19. $f(x) = \dfrac{1}{x^3 - x^2 - 2x}$

20. $f(x) = \dfrac{1}{x^4 - 2x^2 + 1}$

The next group of problems have zeroes as well as poles. So they do intersect the x-axis. Some of the problems have horizontal asymptotes.

21. $f(x) = \dfrac{x - 1}{x - 2}$

22. $f(x) = \dfrac{x - 3}{x + 1}$

23. $f(x) = \dfrac{x - 1}{2 - x}$

24. $f(x) = \dfrac{x + 2}{x}$

25. $f(x) = \dfrac{(x - 1)^2}{x + 1}$

26. $f(x) = \dfrac{x - 1}{(x + 1)^2}$

27. $f(x) = \dfrac{(2x - 2)^2}{(3x + 2)^2}$

28. $f(x) = \dfrac{(6x - 4)^2}{(4x + 5)^2}$

29. $f(x) = \dfrac{x - 1}{(x + 1)(x + 2)}$

30. $f(x) = \dfrac{x + 3}{(x - 1)(x + 2)}$

31. $f(x) = \dfrac{x - 1}{x(x - 3)}$

32. $f(x) = \dfrac{x + 1}{x(x - 3)}$

33. $f(x) = \dfrac{x - 2}{x^2(x + 1)}$

34. $f(x) = \dfrac{x + 2}{x(x - 3)^2}$

35. $f(x) = \dfrac{(x + 2)^2}{(x + 3)(x - 3)}$

36. $f(x) = \dfrac{(x - 2)(x + 3)}{(x - 3)(x + 4)}$

37. $f(x) = \dfrac{(x = 1)(x - 4)}{(x - 2)(x - 3)}$

38. $f(x) = \dfrac{(x - 1)(x - 3)}{(x - 2)(x - 4)}$

39. $f(x) = \dfrac{(x - 3)^2(2x - 4)^3}{(x + 3)^2}$

40. $f(x) = \dfrac{(3x + 1)^3}{(x - 2)^2}$

41. $f(x) = \dfrac{(6x + 4)^3(2x - 3)}{(x + 2)^4}$

42. $f(x) = \dfrac{(x - 3)(x + 4)^2}{(3 - 2x)^2(x + 1)}$

43. $f(x) = \dfrac{(x - 1)(x - 2)(x - 4)^3}{x^2(x - 3)^2}$

44. $f(x) = \dfrac{x^2 + 3x}{(x + 3)^2(x - 4)}$

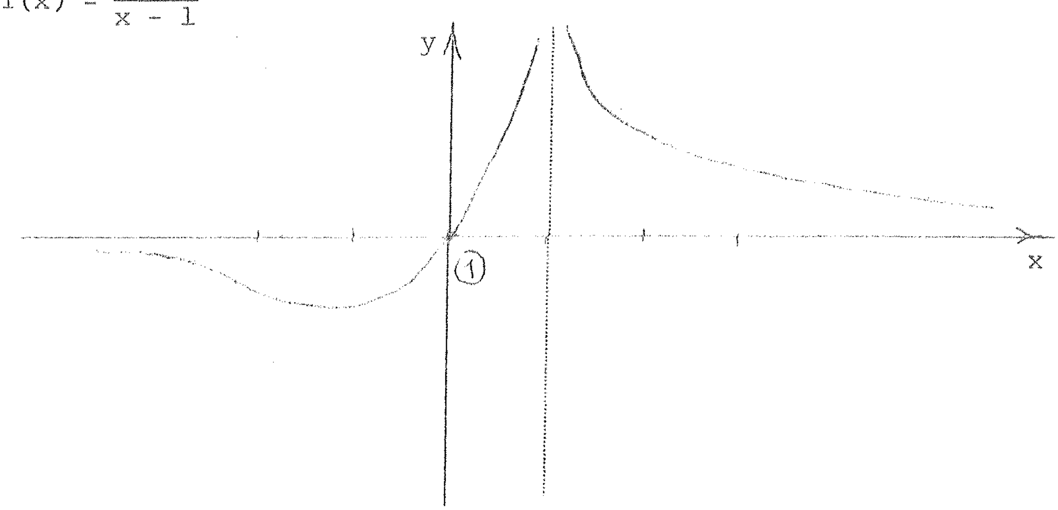
Part II

1. Carefully graph all of the BASIC GRAPHS II, p. 26, on the same set of coordinate axes (colored pencils would help). Plot the points on the graph for the following values of x: ± 2, $\pm 3/2$, ± 1, $\pm 1/2$, $\pm 1/4$. Observe how the graphs are related to one another.

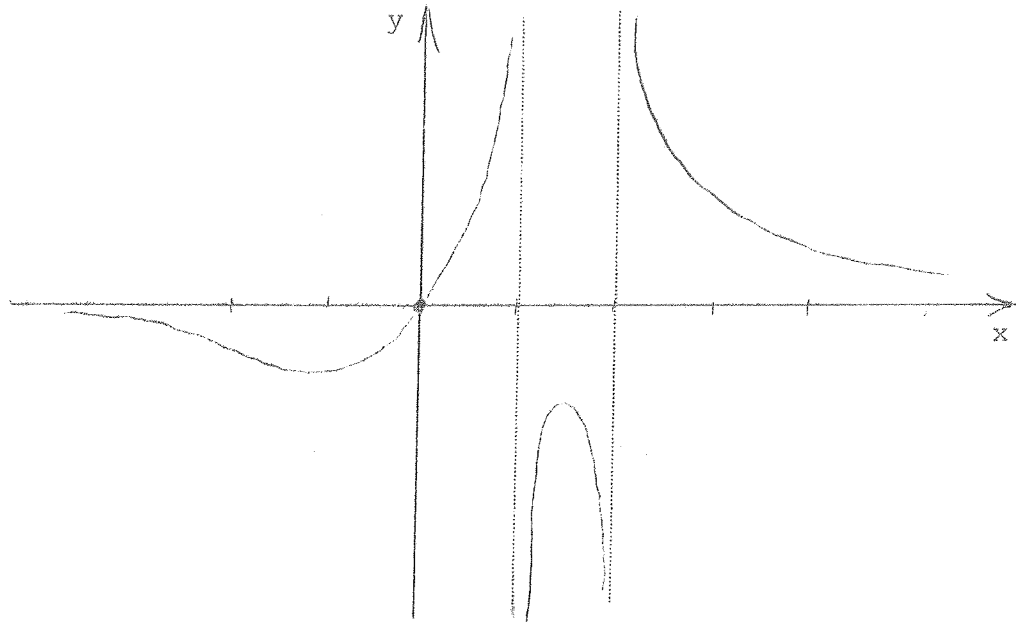
2. On the same coordinate axes plot $y = 1/x^2$, $y = 1/(x - 3)^2$ and $y = -1/x^2$. Plot points for values of $x = \pm 2, \pm 3/2, \pm 1, \pm 1/2$ for the first and third; for $x = 1, 3/2, 2, 5/2, 7/2, 4, 9/2, 5$ for the second. Compare the graphs.

3. What is wrong with each of the following graphs?

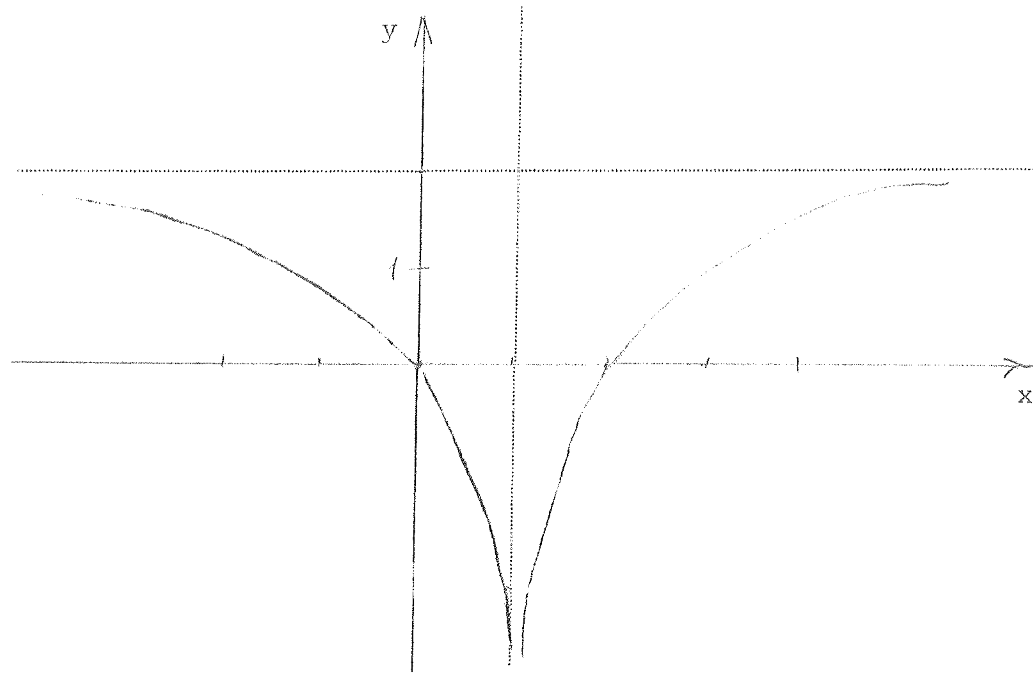
   a) $f(x) = \dfrac{x}{x - 1}$



b) $f(x) = \dfrac{x}{(x - 1)(x - 2)}$

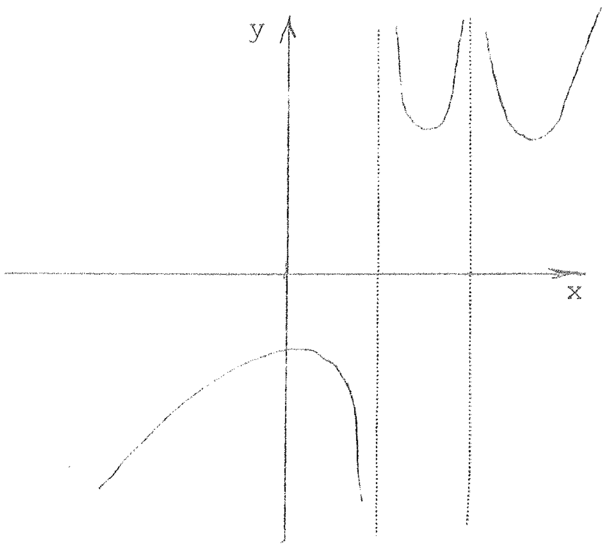


c) $f(x) = \dfrac{x(x + 1)}{(2x - 1)^2}$

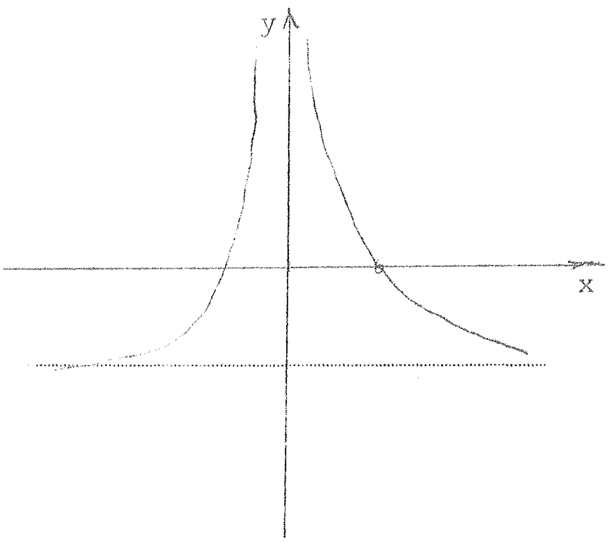


4. Find the rational functions which best fit the following graphs.

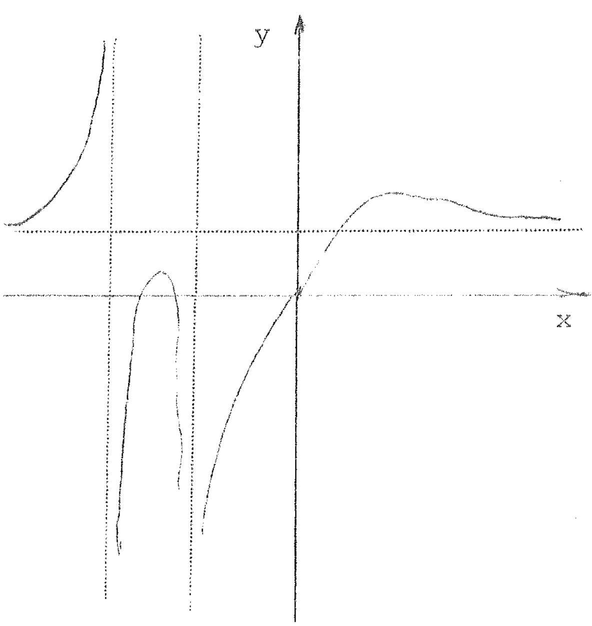
a)



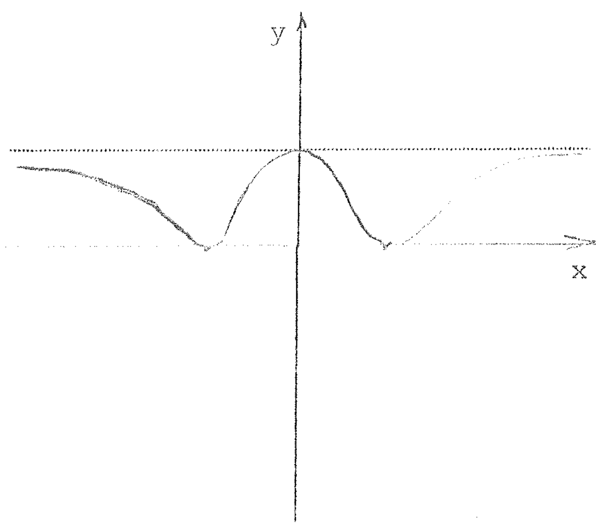
b)



c)



d)



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October 4, 1976